In the following, $U$, $V$, and $W$ are finite dimensional vector spaces over a field $F$.

1. Let $T \in L(V)$ and let $U_1$ and $U_2$ be two $T$-invariant subspaces of $V$.
   (a) Prove that $U_1 + U_2$ is $T$-invariant.
   (b) Prove that $U_1 \cap U_2$ is $T$-invariant.

2. Suppose that $S, T \in L(V)$ are such that $ST = TS$. Prove that null $S$ and range $S$ are $T$-invariant.

3. Suppose that $V$ is a finite dimensional complex vector space and $T \in L(V)$. Prove that $T$ has an invariant subspace of dimension $k$ for each $k = 1, \ldots, \dim V$.

4. Let $n \in \mathbb{Z}_+$ be a positive integer and $T \in L(\mathbb{F}^n)$ be defined by
   
   
   $T(x_1, \ldots, x_n) = (x_1 + \ldots + x_n, \ldots, x_1 + \ldots + x_n)$

   for every $x_1, x_2, \ldots, x_n \in \mathbb{F}$. Compute the eigenvalues and associated eigenvectors for $T$.

5. Suppose that $V = U \oplus W$, where $U$ and $W$ are subspaces of $V$. Define $P \in L(V)$ by
   
   $P(u + w) = u$ for every $u \in U$ and $w \in W$. Find all eigenvalues and eigenvectors of $P$.

6. Suppose $T \in L(V)$ and $S \in L(V)$ invertible. Show that $T$ and $S^{-1}TS$ have the same eigenvalues.

7. Suppose that $T \in L(V)$ is invertible. Prove that
   
   $\text{null}(T - \lambda I) = \text{null}(T^{-1} - \frac{1}{\lambda} I)$

   for every $\lambda \in \mathbb{F}$ with $\lambda \neq 0$.

8. Suppose that $T \in L(V)$ has the property that every $v \in V$ is an eigenvector for $T$. Prove that $T$ must then be a scaler multiple of the identity function on $V$.

9. Let $V$ be a finite dimensional over $\mathbb{C}$, $T \in L(V)$, and $p(z) \in \mathbb{C}[z]$ be a polynomial. Prove that $\lambda \in \mathbb{C}$ is an eigenvalue of the linear operator $p(T) \in L(V)$ if and only if $T$ has an eigenvalue $\mu \in \mathbb{C}$ such that $p(\mu) = \lambda$. 