In all of the following questions, $V$ is a finite dimensional vector space over a field $F$.

1. [15 points] Solve the equation $z^3 - 4i = 0$, where $z \in \mathbb{C}$.

2. [10 points] Determine whether the following set of vectors are linearly independent in $\mathbb{R}^3$.

$$v_1 = (1, 1, 1), \quad v_2 = (0, 1, -1), \quad v_3 = (1, 1, 0)$$
3. [15 points]
   (a) List all subspaces of \( \mathbb{R} \), \( \mathbb{R}^2 \), and \( \mathbb{R}^3 \).
   (b) Give an example of a nonempty subset of \( \mathbb{R}^2 \) that is closed under scalar multiplication but it is not a subspace of \( \mathbb{R}^2 \).
   (c) Give an example of a nonempty subset of \( \mathbb{R}^2 \) that is closed under vector addition but it is not a subspace of \( \mathbb{R}^2 \).
4. [20 points] Let $V_1$ and $V_2$ be two subspaces of $V$,
   (a) Prove that $V_1 \cap V_2$ is a subspace of $V$.
   (b) Prove that $V_1 + V_2$ is a subspace of $V$.
   (c) Give examples of $V$, and subspaces $V_1$ and $V_2$ of $V$ in each of the following cases
       i. $V_1 \cup V_2$ is a subspace of $V$.
       ii. $V_1 \cup V_2$ is not a subspace of $V$. 
5. [10 points] Let \( v_j \in V \) for \( j = 1, 2, \ldots, n \). Write a mathematical definition of each of the following:

(a) The list of vectors \((v_1, v_2, \ldots, v_n)\) is linearly dependent.

(b) \( V = \text{span}(v_1, v_2, \ldots, v_n) \).

6. [20 points] Let \( W_1 \) and \( W_2 \) be two subspaces of \( V \), and suppose that \( V = W_1 + W_2 \), prove that \( V = W_1 \oplus W_2 \) if and only if \( W_1 \cap W_2 = \{0\} \).
7. [20 points] Suppose that \((v_1, v_2, \ldots, v_n)\) be a linearly independent list of vectors in \(V\). Given any \(w \in V\) such that \((v_1 + w, v_2 + w, \ldots, v_n + w)\) is a linearly dependent list of vectors in \(V\), prove that \(w \in \text{span}(v_1, v_2, \ldots, v_n)\).
8. [10 points] Suppose that $W_1$ and $W_2$ are two subspaces of $V$ such that $W_1 \cap W_2 = \{0\}$. Let $(v_1, v_2, \ldots, v_m)$ and $(w_1, w_2, \ldots, w_n)$ be two linearly independent sets in the subspaces $W_1$ and $W_2$, respectively. Prove that

$$(v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_m)$$

is linearly independent in $W_1 \oplus W_2$. 