(1) Let $V$ be a finite dimensional vector space over the field $\mathbb{F}$, and suppose that $P \in \mathcal{L}(V)$ has the properties that $P^2 = P$. Prove that $V = \text{null}(P) \oplus \text{range}(P)$.

(2) Solve for $x$ the following equation
\[
\det \begin{bmatrix} x & -1 \\ 3 & 1 - x \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & -3 \\ 2 & x & -6 \\ 1 & 3 & x - 5 \end{bmatrix}
\]

(3) Use the definition of the determinant to derive the formula for the determinant of a $4 \times 4$ matrix.

(4) Find the determinant of the following matrix
\[
\begin{bmatrix} \sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta) & \sin(\theta) & 0 \\ \sin(\theta) - \cos(\theta) & \sin(\theta) + \cos(\theta) & 1 \end{bmatrix}
\]

(5) Let $A$ be an invertible matrix in $\mathbb{R}^{n \times n}$, suppose that $A^2 - 2A = 0$, find $\det(2A^T A^2)$.

(6) Suppose $(V, \langle \cdot, \cdot \rangle)$ is an inner product space. Let $T \in \mathcal{L}(V)$ is injective. Define $\langle \cdot, \cdot \rangle_1$ by
\[
\langle u, v \rangle_1 := \langle Tu, Tv \rangle, \text{ for all } u, v \in V.
\]
Show that $\langle \cdot, \cdot \rangle_1$ is an inner product on $V$.

(7) Suppose that $V$ is a real inner product space
(a) Show that if $u, v \in V$ have the same norm, then $u + v$ is orthogonal to $u - v$.
(b) Use part (a) to show that the diagonals of a rhombus are perpendicular to each other.

(8) Let $V$ be a finite dimensional inner product space over $\mathbb{R}$. Given $u, v \in V$, prove that
\[
\langle u, v \rangle = \frac{1}{4} \left( \|u + v\|^2 - \|u - v\|^2 \right).
\]

(9) Suppose the $T \in \mathcal{L}(V)$ is such that $\|Tv\| \leq \|v\|$ for all $v \in V$. Prove that $(T - 2\mathbb{1})$ is invertible.