• Submit questions 2 and 5 for grading.
• Due on: Tuesday June 25, 2019.
• You will have a quiz from the following questions on Tuesday June 25, 2019.

(1) Suppose \((V_1, \langle \cdot, \cdot \rangle_1), \ldots, (V_n, \langle \cdot, \cdot \rangle_n)\) are inner product spaces. Show that
\[
\langle (u_1, \ldots, u_n), (v_1, \ldots, v_n) \rangle := \langle u_1, v_1 \rangle_1 + \ldots + \langle u_n, v_n \rangle_n
\]
defines an inner product on \(V_1 \times \ldots \times V_n\).

(2) Let \((e_1, e_2, e_3)\) be the canonical basis of \(\mathbb{R}^3\), and define
\[f_1 = e_1 + e_2 + e_3, \quad f_2 = e_2 + e_3, \quad f_3 = e_3\]
(a) Apply the Gram-Schmidt process to the basis \((f_1, f_2, f_3)\).
(b) What do you obtain if you instead applied the Gram-Schmidt process to the basis \((f_3, f_2, f_1)\)?)

(3) Let \(\mathbb{R}_2[x]\) be the inner product space of polynomials over \(\mathbb{R}\) having degree at most two, with inner product given by
\[
\langle f, g \rangle = \int_0^1 f(x)g(x) \, dx, \text{ for every } f, g \in \mathbb{R}_2[x].
\]
Apply the Gram-Schmidt procedure to the standard basis \(\{1, x, x^2\}\) for \(\mathbb{R}_2[x]\) in order to produce an orthonormal basis for \(\mathbb{R}_2[x]\).

(4) Prove that
\[
16 \leq (a + b + c + d) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right)
\]
for all positive numbers \(a, b, c, d\).

(5) Let \(n \in \mathbb{Z}_+\), and let \(a_1, a_2, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}\) be any collection of \(2n\) real numbers. Prove that
\[
\left( \sum_{k=1}^n a_k b_k \right)^2 \leq \left( \sum_{k=1}^n k a_k^2 \right) \left( \sum_{k=1}^n \frac{b_k^2}{k} \right).
\]

(6) Let \(V\) be a finite dimensional vector inner product space over \(F\), and suppose that \(P \in \mathcal{L}(V)\) with \(P^2 = P\) and \(\text{null}(P) = (\text{range}(P))^\perp\). Prove that \(P\) is an orthogonal projection.
Hint: compare and use Question 1 from Hw 7.

(7) Suppose \(V\) is finite dimensional and \(U\) is a subspace of \(V\). Let \(P_U\) be the orthogonal projection onto \(U\). Show that
\[
P_{U^\perp} = \mathbb{1} - P_U
\]
where \(\mathbb{1} \in \mathcal{L}(V)\) is the identity map on \(V\).