Quenched correlations in disordered harmonic oscillator systems

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Based on a joint work with R. Sims, and G. Stolz.

Great Lakes Mathematical Physics 2017

June 23, 2017
The Harmonic Oscillators

**The Hamiltonian**

\[ H_{\Lambda} = \sum_{x \in \Lambda} (p_x^2 + k_x q_x^2) + \sum_{\{x, y\} \subset \Lambda, |x - y| = 1} \lambda(q_x - q_y)^2 \]

- \( \Lambda := ([a_1, b_1] \times \ldots \times [a_d, b_d]) \cap \mathbb{Z}^d \) for integers \( a_j < b_j \) for all \( j \), and \( d \geq 1 \).
- For each \( x \in \Lambda \), \( q_x \) and \( p_x = -i \frac{\partial}{\partial q_x} \) are the position and momentum operators.
- The Hilbert space \( \mathcal{H}_{\Lambda} = \bigotimes_{x \in \Lambda} \mathcal{L}^2(\mathbb{R}) = \mathcal{L}^2(\mathbb{R}^\Lambda) \).
- \( \lambda \in (0, \infty) \) is the coupling parameter.
- \( \{k_x\}_x \) are i.i.d. random variables with absolutely continuous distribution given by a bounded density \( \nu \) supported in \([0, k_{\text{max}}]\).
Correlations of the Harmonic Oscillators
Known Results and the New Contribution

**Known:** Exponential decay of the position-momentum correlations at the:
- ground state
- thermal states.

Nachtergaele-Sims-Stolz (2012).

**New Results:**
1. Correlations at the energy eigenstates.
2. Correlations after a quantum quench.
**Diagonalizing** $H_\Lambda$

- Define the *annihilation* and *creation* operators

\[ a_x = \frac{1}{\sqrt{2}}(q_x + ip_x), \quad a_x^* = \frac{1}{\sqrt{2}}(q_x - ip_x), \quad \text{for} \; x \in \Lambda. \]

They satisfy the *Canonical Commutations Relations (CCR)*

\[ [a_x, a_y] = [a_x^*, a_y^*] = 0, \quad \text{and} \quad [a_x, a_y^*] = \delta_{x,y} 1, \quad \text{for all} \; x, y \in \Lambda. \]

- The harmonic Hamiltonian can be written as

\[ H_\Lambda = \frac{1}{2} \left( a^T \quad (a^*)^T \right) \begin{pmatrix} h_\Lambda - 1 & h_\Lambda + 1 \\ h_\Lambda + 1 & h_\Lambda - 1 \end{pmatrix} \begin{pmatrix} a \\ a^* \end{pmatrix}. \]

- $h_\Lambda$ is the finite volume Anderson model on $\ell^2(\Lambda)$, i.e.,

\[ h_\Lambda = \lambda h_{0,\Lambda} + k \]

where $h_{0,\Lambda}$ is the discrete Laplacian operator, and

\[ k := \text{diag}\{k_x, \; x \in \Lambda\}. \]
Diagonalizing $H_\Lambda$

- $\sigma(h_\Lambda) \subset \left[ \min_{x \in \Lambda} k_x, 4d\lambda + k_{\max} \right]$.
- $h_\Lambda$ is diagonalizable in terms of an orthogonal matrix $O$ and a diagonal operator $\gamma^2 = \text{diag}\{\gamma^2_x, x \in \Lambda\}$, i.e., $h_\Lambda = O\gamma^2 O^T$.
- Define the bosons $\{b_k, k = 1, \ldots, |\Lambda|\}$ using the Bogoliubov transformation

$$b = U a + V a^*$$

where

$$U = \frac{1}{2}(\gamma \frac{1}{2} + \gamma^{-\frac{1}{2}})O^T, \quad V = \frac{1}{2}(\gamma \frac{1}{2} - \gamma^{-\frac{1}{2}})O^T.$$ 

- $H_\Lambda$ can be written as a free boson system

$$H_\Lambda = \sum_{k=1}^{\mid\Lambda\mid} \gamma_k (2b_k^*b_k + \mathbb{1})$$
A free boson system

\[ H_{\Lambda} = \sum_{k=1}^{|\Lambda|} \gamma_k (2b^*_k b_k + 1) \] ← Free boson system.

- The operators \( b_k \) satisfy the CCR
- \( \{\gamma_k > 0, k = 1, \ldots, |\Lambda|\} \) are the eigenvalues of \( h_{\Lambda}^{\frac{1}{2}} \).
- The eigen-pair of \( H_{\Lambda} \) associated with \( \alpha = (\alpha_1, \ldots, \alpha_{|\Lambda|}) \in \mathbb{N}_0^{\Lambda} \) is \((\psi_\alpha, E_\alpha)\),

\[
\psi_\alpha = \prod_{j=1}^{\Lambda} \frac{1}{\sqrt{\alpha_j}!} (b_j^*)^{\alpha_j} |\text{vac}_b\rangle, \quad E_\alpha = \sum_{j=1}^{\Lambda} (2\alpha_j + 1) \gamma_j
\]
The Harmonic Oscillators

The Eigencorrelator Localization

**Assumption:** The eigencorrelator localization

There exist constants $C < \infty$ and $\eta > 0$ and $0 < s \leq 1$, independent of $\Lambda$, such that

$$
\mathbb{E} \left( \sup_{|g| \leq 1} |\langle \delta_x, h^{-\frac{1}{2}} \Lambda g(h) \delta_y \rangle|^s \right) < Ce^{-\eta|x-y|},
$$

for all $x, y \in \Lambda$.

Satisfied for

- $d \geq 1$: large disordered case with $s = 1$.
- $d = 1$: any distribution density $\nu$ with $s = \frac{1}{2}$.
The Harmonic Oscillators

Correlation Matrix

- For any observable $A$ and state $\rho$, $\langle A \rangle_\rho := \text{Tr}[A \rho]$.
- The position-position dynamical correlation at state $\rho$ is
  \[ \langle \tau_t(p_x)p_y \rangle_\rho - \langle \tau_t(p_x) \rangle_\rho \langle p_y \rangle_\rho, \quad x, y \in \Lambda. \]
- $\tau_t(q_x) = e^{itH_\Lambda} q_x e^{-itH_\Lambda}$.
- We will consider states $\rho$ such that $\langle \tau_t(p_x) \rangle_\rho = \langle \tau_t(q_x) \rangle_\rho = 0$ for all $x \in \Lambda$ and $t \geq 0$.
- Define the positions-momenta correlation matrix
  \[ \Gamma_\rho(t) := \begin{pmatrix} \langle \tau_t(q)q^T \rangle_\rho & \langle \tau_t(q)p^T \rangle_\rho \\ \langle \tau_t(p)q^T \rangle_\rho & \langle \tau_t(p)p^T \rangle_\rho \end{pmatrix} \]
- Let
  \[ (\Gamma_\rho(t))_{xy} = \begin{pmatrix} \langle \tau_t(q_x)q_y \rangle_\rho & \langle \tau_t(q_x)p_y \rangle_\rho \\ \langle \tau_t(p_x)q_y \rangle_\rho & \langle \tau_t(p_x)p_y \rangle_\rho \end{pmatrix} \]
Eigenstates Correlations

Recall

The eigencorrelator localization assumption: There exist constants $C < \infty$ and $\eta > 0$ and $0 < s \leq 1$, independent of $\Lambda$, such that

$$\mathbb{E} \left( \sup_{|g| \leq 1} |\langle \delta_x, h^{-\frac{1}{2}}g(h)\delta_y \rangle|^s \right) < C e^{-\eta|x-y|}, \text{ for all } x, y \in \Lambda. \quad (1)$$

Theorem

Under the eigencorrelator localization assumption (above),

$$\mathbb{E} \left( \sup_t \| (\Gamma_{\rho_\alpha}(t))_{xy} \|^s \right) \leq C C' (1 + \|\alpha\|_{\infty})^{1+s} e^{-\eta|x-y|}$$

for all finite rectangular boxes $\Lambda \subset \mathbb{Z}^d$, $x, y \in \Lambda$ and $\alpha \in \mathbb{N}_0^{\Lambda}$. Here $C$, $\eta$ and $s$ are as in (1) and $C' < \infty$ depends on $d$, $\lambda$, $s$ and $k_{\text{max}}$, but is independent of $\Lambda$. 
Decompose $\Lambda$ into $M$ disjoint rectangular sub-boxes, $\Lambda = \bigcup_{\ell=1}^{M} \Lambda_\ell$.

For each $\ell$, let $H_{\Lambda_\ell}$ be the restriction of $H_\Lambda$ to $\Lambda_\ell$.

Let $H_{0,\Lambda}$ be the Hamiltonian of the non-interacting system on $\mathcal{H}_\Lambda$, 

$$H_{0, \Lambda} = \sum_{\ell=1}^{M} H_{\Lambda_\ell} \otimes 1_{\Lambda \setminus \Lambda_\ell}.$$ 

Let $\{\rho_\ell, \ell = 1, \ldots, M\}$ be states acting on $L^2(\mathbb{R}^{\Lambda_\ell})$, and let 

$$\rho := \bigotimes_{\ell=1}^{M} \rho_\ell.$$ 

We study the positions-momenta correlations at the state 

$$\rho_t = e^{-itH_\Lambda} \rho e^{itH_\Lambda}.$$
Recall that $\rho_t = e^{-itH_\Lambda} \left( \bigotimes_{\ell=1}^{M} \rho_\ell \right) e^{itH_\Lambda}$.

For every $x, y \in \Lambda$, let

$$(\Gamma_{\rho_t})_{xy} := (\Gamma_{\rho_t(0)})_{xy} = \begin{pmatrix} \langle q_x q_y \rangle_{\rho_t} & \langle q_x p_y \rangle_{\rho_t} \\ \langle p_x q_y \rangle_{\rho_t} & \langle p_x p_y \rangle_{\rho_t} \end{pmatrix}$$

For all $x, y \in \Lambda_\ell$

$$(\Gamma_{\rho_\ell})_{x,y} := (\Gamma_{\rho_\ell(0)})_{xy} = \begin{pmatrix} \langle q_x q_y \rangle_{\rho_\ell} & \langle q_x p_y \rangle_{\rho_\ell} \\ \langle p_x q_y \rangle_{\rho_\ell} & \langle p_x p_y \rangle_{\rho_\ell} \end{pmatrix}$$
Recall: The Eigencorrelator Localization Assumption,
\[
\mathbb{E} \left( \sup_{|g| \leq 1} |\langle \delta_x, h^{-\frac{1}{2}} g(h) \delta_y \rangle|^s \right) < C e^{-\eta|x-y|}, \text{ for all } x, y \in \Lambda.
\]  
(2)

**Theorem**

Under the assumption given above. Suppose that, for some \( C' < \infty \), and \( \eta' > 0 \),
\[
\mathbb{E} \left( \| (\Gamma_{\rho_{\ell}})_{xy} \|^s \right) \leq C' e^{-\eta'|x-y|}
\]  
(3)

for all \( \ell \) and all \( x, y \in \Lambda_{\ell} \), where \( 0 < s \leq 1 \) is as in (2).

Then, for \( \eta \) from (2), \( \tilde{\eta} := \frac{1}{6} \min\{\eta, \eta'\} \) and \( \rho = \bigotimes_{\ell} \rho_{\ell} \), there exists a constant \( C''' < \infty \) such that
\[
\mathbb{E} \left( \sup_{t \in \mathbb{R}} \| (\Gamma_{\rho_t})_{xy} \|^s \right) \leq (C')^{1/3} C''' e^{-\tilde{\eta}|x-y|}
\]

for all \( x, y \in \Lambda \). Here \( C' \) is the constant from (3) and \( C''' \) depends on \( d, \lambda, s, k_{\max} \) and \( \tilde{\eta} \), but is independent of \( \Lambda \) and the number of subregions \( M \).
Correlations After a Quantum Quench

Thermal States Correlations

Thermal States:

\[ \rho_\beta = \frac{e^{-\beta H_\Lambda}}{\text{Tr}[e^{-\beta H_\Lambda}]}, \text{ for } \beta \in (0, \infty). \]

**Theorem**

For a rectangular box \( \Lambda \subset \mathbb{Z}^d \) and \( \beta \in (0, \infty) \), let \( \Gamma_{\rho_\beta} = \Gamma_{\rho_\beta}(0) \) their static position-momentum correlation matrices.

There exist \( C < \infty \) and \( \mu > 0 \), dependent on \( d, \lambda \) and the distribution of the random variables \( k_x \), but independent of \( \Lambda \) and \( \beta \), such that

\[ \mathbb{E} \left( \| (\Gamma_{\rho_\beta})_{xy} \|^{\frac{1}{2}} \right) \leq C \max \left\{ 1, \frac{1}{\beta} \right\} e^{-\mu |x-y|} \]

for all \( x, y \in \Lambda \).
Consider the thermal states of $H_{\Lambda_\ell}$ with inverse temperatures $\beta_\ell$, $\ell = 1, \ldots, M$, i.e.,

$$
\rho_{\ell, \beta_\ell} = \frac{e^{-\beta_\ell H_{\Lambda_\ell}}}{\text{Tr}[e^{-\beta_\ell H_{\Lambda_\ell}}]}.
$$

The product state

$$
\rho_{\beta_1, \ldots, \beta_M} := \bigotimes_{\ell=1}^M \rho_{\ell, \beta_\ell}.
$$

The Schrödinger evolution

$$
(rho_{beta_1, \ldots, beta_M})_t = e^{-itH_{\Lambda}}(rho_{beta_1, \ldots, beta_M})e^{itH_{\Lambda}}.
$$

We assume the eigencorrelator localization with $s = 1/2$.

**Result:**

$$
\mathbb{E} \left( \sup_t \| (\Gamma(rho_{beta_1, \ldots, beta_M})_t)_{xy} \|^2 \right)^{1/6} \leq C' \max \left\{ 1, \beta^{-1/3} \right\} e^{-\tilde{\eta}|x-y|}
$$
Correlations After a Quantum Quench

Corollaries: Energy Eigenstates

For $\ell = 1, \ldots, M$, let $\alpha_\ell \in \mathbb{N}_{0}^{\Lambda_\ell}$, and $\rho_{\alpha_\ell}$ be the corresponding “local” eigenstate of $H_{\Lambda_\ell}$.

Let $N$ be the highest mode, $\|\alpha_\ell\|_\infty \leq N$ for all $\ell = 1, \ldots, M$.

The product state

$$\rho_{\alpha} = \bigotimes_{\ell=1}^{M} \rho_{\alpha_\ell}.$$  

The time evolution

$$(\rho_{\alpha})_t := e^{-itH_{\Lambda}} \rho_{\alpha} e^{itH_{\Lambda}}.$$  

We assume the eigencorrorralator localization with $s = \frac{1}{2}$.

Result:

$$\mathbb{E} \left( \sup_t \| (\Gamma_{(\rho_{\alpha})_t})_{xy} \|_{\frac{1}{6}} \right) \leq \tilde{C} (1 + N)^{\frac{1}{2}} e^{-\frac{n}{6} |x-y|}$$

for all $x, y \in \Lambda$. 

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Correlations After a Quantum Quench

**Corollaries: Eigenstates-Thermal states**

- Fix $\beta > 0$ and $N < \infty$.
- Consider the local states $\rho_\ell$, $\ell = 1, \ldots, M$, where each $\rho_\ell$ is one of the following:
  - a thermal state with inverse temperature $\beta_\ell \in (\beta, \infty)$.
  - an eigenstate with occupation number vector $\alpha_\ell$ with $\|\alpha_\ell\|_\infty \leq N$.
- Let $\rho = \bigotimes_{\ell=1}^{M} \rho_\ell$ and $\rho_t = e^{-itH_\Lambda} \left( \bigotimes_{\ell=1}^{M} \rho_\ell \right) e^{-itH_\Lambda}$.
- We assume the eigencorrelator localization with $s = 1/2$.

**Result:**

\[
\mathbb{E} \left( \sup_t \| (\Gamma_{\rho_t})_{xy} \|^{\frac{1}{6}} \right) \leq C \max \left\{ (1 + N)^{\frac{3}{2}}, \frac{1}{\beta} \right\}^{\frac{1}{3}} e^{-\tilde{\eta}|x-y|}
\]

for all $x, y \in \Lambda$. 
Correlations After a Quantum Quench

Corollaries: \#decompositions = The Volume of the system

- If the $M = |\Lambda|$.
- $\{H\{x\}, x \in \Lambda\}$ with $H\{x\} = p_x^2 + k_x q_x^2$.
- Let $\{n_x, x \in \Lambda\}$ be the set of occupation numbers in sites $x \in \Lambda$.
- Let $N = \max_x n_x$, i.e., the maximum occupation number.
- The eigenstates are

$$\phi_{n_x}(q_x) = \text{Const.} \cdot H_{n_x}(\sqrt[k_x]{q_x})e^{-\frac{\sqrt{k_x}}{2}q_x^2}, \text{ for } x \in \Lambda.$$ 

- Let $\rho = \bigotimes_{x \in \Lambda} \rho_x$ and $\rho_t = e^{-itH_{\Lambda}} \left( \bigotimes_{x \in \Lambda} \rho_x \right) e^{itH_{\Lambda}}$.
- We assume eigencorrelator localization with $s = 1/2$.

Result:

$$\mathbb{E} \left( \sup_t \| (\Gamma_{\rho_t})_{xy} \|^{\frac{1}{6}} \right) \leq C(1 + 2N)^{\frac{1}{6}} e^{-\frac{n}{6}|x-y|}$$

for all $x, y \in \Lambda$. 
Thank you.