MATH 574M: Homework 4 Solution

1. Read the paper and write the summary.

2. Ex 2.6 in the textbook.

Assume there are totally $N$ unique value of $x$’s, which are denoted as $z_j$. For each $z_j$, we assume that there are $n_j \geq 1$ observations and define $N_j = \{i : x_i = z_j, i = 1, \cdots, n\}$ for $j = 1, \cdots, N$. Then $\sum_{j=1}^{N} n_j = n$ The ordinary least squares (OLS) minimizes the following problem

$$\hat{\theta}_{LS} = \arg \min_{\theta} \sum_{i=1}^{n} [y_i - f_\theta(x_i)]^2$$

$$= \arg \min_{\theta} \sum_{j=1}^{N} \sum_{i \in N_j} [y_i - f_\theta(z_j)]^2$$

$$= \arg \min_{\theta} \sum_{j=1}^{N} [N_j f_\theta^2(z_j) - 2 N_j f_\theta(z_j) \sum_{i \in N_j} y_i]$$

$$= \arg \min_{\theta} \sum_{j=1}^{N} N_j [f_\theta(z_j) - \bar{y}_j]^2, \quad \bar{y}_j = \sum_{i \in N_j} y_i / N_j.$$

So the OLS for this case is equivalent to the weighted least squares problem associated with the data $(z_j, \bar{y}_j), j = 1, \cdots, N$ and the weights $N_j, j = 1, \cdots, N$.

3. Exercise 2.7 in the textbook.

Solution:

• (a) For linear regression:

$$\hat{f}(x_0) = x_0^T \hat{\beta} = x_0^T (X^T X)^{-1} X^T Y$$

So $l_i(x_0; \mathcal{X}) = x_0^T (X^T X)^{-1} x_i$.

For K-nearest-neighbor regression:

$$\hat{F}(x_0) = \frac{1}{k} \sum_{x_i \in N_k(x_0)} y_i$$

So $l_i(x_0; \mathcal{X}) = 1/k$ if $x_i \in N_k(x_0), 0$ otherwise.
(b) Begin with the familiar bias-variance decomposition:

$$E_{Y\mid X}(f(x_0) - \hat{f}(x_0))^2 = E_{Y\mid X}(f(x_0) - E_{Y\mid X}\hat{f}(x_0))^2 + E_{Y\mid X}(E_{Y\mid X}\hat{f}(x_0) - \hat{f}(x_0))^2$$

$$= Bias^2_{Y\mid X}(\hat{f}(x_0)) + Var_{Y\mid X}(\hat{f}(x_0))$$

Now, we express the bias and variance in terms of $l_i$.

$$Bias_{Y\mid X}(\hat{f}(x_0)) = E_{Y\mid X}\left(\sum_{i=1}^{N} l_i(x_0; \mathcal{X})y_i - f(x_0)\right)$$

$$= \sum_{i=1}^{N} l_i(x_0; \mathcal{X})f(x_i) - f(x_0)$$

$$Var_{Y\mid X}(\hat{f}(x_0)) = Var_{Y\mid X}\left(\sum_{i=1}^{N} l_i(x_0; \mathcal{X})y_i\right)$$

$$= \sigma^2 \sum_{i=1}^{N} l_i^2(x_0; \mathcal{X})$$

4. For Questions 4-6, please see the code.