1 Contents

2 Introduction

Many interesting problems of nonlinear wave propagation arise from the process of interaction of intense electromagnetic waves with different types of matter. Numerous models of atomic medium were introduced to describe different phenomena related to this interaction. In particular, if the frequency of the electromagnetic wave is equal to the transition frequency between some atomic levels in the medium, then there might occur a strong resonant interaction. Self-induced transparency, superflourescence, light amplification and photon echo are known to appear as a consequence of such behavior and are all described by the Maxwell-Bloch equations which are considered in the present work.

In the theoretical description of this resonance it is often possible to neglect existence of all other atomic energy levels and consider interaction of light with the so-called two-level medium. Quantum theory of the two-level atom is relatively simple and consideration of the electromagnetic wave in the framework of classical theory provides additional simplification. Such assumptions are justified by studies of almost all phenomena of interaction of intense electromagnetic beams with atoms. Equations of this theoretical model are of significant mathematical interest on their own. They represent an example of completely integrable infinite-dimensional Hamiltonian system which contains enormously rich underlying structure. Mathematical techniques for analysis of such equations interconnect seemingly irrelevant fields and are of a very close attention for last few decades.

In the introduction we shall shortly describe derivation of equations of the model, futher we shall discuss mathematical problems which arise from physically interesting phenomena and inverse scattering method, which is the widely known tool for their solution. However inverse scattering is only suitable for Cauchy problems, i. e. for problems on the infinite or periodic domain, where initial condition is prescribed. Most of the physical problems are in fact initial-boundary-value problems, e. g. initial state of the medium is pescribed, but incoming electromagnetic field is only given on one boundary of the medium as a function of time. As the main contribution of this work we shall present different method for addressing initial-boundary-value problems for these equations and as an example – solution of the problem of quantum amplifier in Born approximation.

There exists a set of works devoted to study of MB equations. This system first appeared in works of Lamb [1,2] who introduced it for the description of resonant interaction of light with matter. It was first integrated using inverse scattering transformation (IST) by Anlowits, Kaup and Newell [3]. Extensive research of these equations was conducted by Gabitov, Zakharov and Mikhailov [4,5], who exploited the same IST method to study the problem of superflourescence.

2.1 Description of the electromagnetic field

Vectors of electromagnetic field are described by the Maxwell equations

$$\nabla \times \mathcal{E} = -\frac{1}{c}\frac{\partial \mathcal{B}}{\partial t} \tag{1}$$

$$\nabla \times \mathcal{B} = \frac{4\pi}{c} j + \frac{1}{c}\frac{\partial \mathcal{D}}{\partial t} \tag{2}$$

$$\nabla \cdot \mathcal{E} = 4\pi\rho \tag{3}$$

$$\nabla \cdot \mathcal{B} = 0 \tag{4}$$

where $\mathcal{D} = \mathcal{E} + \triangle\pi\mathcal{P}$, $\mathcal{P}$ is polarizaton of the medium.

We shall discuss wave propagation in the electrically neutral medium, hence we may set $\rho = j = 0$. The term of the most importance is then polarization $\mathcal{P}$. Polarization occurs as a consequence of deformation of atomic shape under the influence of electromagnetic field. This interaction is the origin of nonlinearity in the problem. Electromagnetic wave satisfies the wave equation, which may be obtained by taking curl of (1) and using (2).

$$\nabla^2 \mathcal{E} - \frac{1}{c^2}\frac{\partial^2 \mathcal{E}}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 \mathcal{P}}{\partial t^2}. \tag{5}$$

Let us concentrate on propagation of plane, simply-polarized waves. This simplification already provides reasonable agreement with experiments and rich mathematical models. Since the duration of electromagnetic pulse varies from 10^{-9}s to 10^{-12}s, whereas period of oscillations in the wave itself is about 10^{-15}, even the shotest pulses contain many wavelengths. Therefore it is reasonable to represent $\mathcal{E}$ as a rapidly oscillating plane wave with slowly varying amplitude and phase

$$\mathcal{E}(x,t) = E(x,t)\cos[kx - \omega t + \phi(x,t)]. \tag{6}$$

Slow variation of the amplitude means that

$$\frac{\partial E}{\partial t} \ll \omega E, \quad \frac{\partial E}{\partial x} \ll kE. \tag{7}$$

Similar realtions should be true for ϕ as well. Let us now turn to consideration of the interaction of medium with such field.

2.2 Two level atom

As it has already been mentioned, we shall only consider idealized medium, where atoms only have two energy levels. We assume that the energy difference between upper level a and lower level b is approximately equal to the frequency ω_0 of the electromagnetic field. In other words we assume that the resonance condition $E_a - E_b = \hbar\omega_{ab} \approx \hbar\omega_0$, where $\hbar$ is the Plank constant divided by 2π, holds. In this case we can write the wavefunction of the atom as a time-dependent linear combination of normalized wavefunction of of these two levels, i. e.

$$\psi(r,t) = a(t)\psi_a(r) + b(t)\psi_b(r). \tag{8}$$

Normalization of ψ yields $|a|^2 + |b|^2 = 1$. Difference in the population density is equal to

$$n = n_0 \int (|a\psi_a|^2 - |b\psi_b|^2)\mathrm{d}r = n_0(|a|^2 - |b|^2). \tag{9}$$

Wavefunction ψ satisfies Schrödinger equation

$$\hat{H}\psi = \mathrm{i}\hbar\frac{\partial\psi}{\partial t}, \tag{10}$$

where Hamiltonian $\hat{H}$ contains term $-\hbar^2/2m\nabla^2$, describing external motion of the atom, $\hat{H}_0$ – Hamiltonian of the free atom and Hamiltonian of interaction with electric field $\hat{H}_I = -\mathbf{d}\cdot\mathcal{E}$. Here $\mathbf{d}$ is a dipole moment of the atom, $\mathbf{d} = -\mathrm{e}\,\mathbf{r}$, e is elementary charge, r – internal coordinate of the atom. We assume that $\mathbf{d}$ is parallel to $\mathcal{E}$ and therefore $\hat{H}_I = -d\mathcal{E}$. Polarization of the atom may be written as

$$p = \int \psi^* d\psi \mathrm{d}r. \tag{11}$$

If we assume that atoms in the medium do not have their own dipole moment, e. i. $\int |\psi_\alpha|^2 r\mathrm{d}r = 0$, then (11) is reduced to

$$p = p_0(a^*b + b^*a), \tag{12}$$

where

$$p_0 = -\mathrm{e}\int \psi_a^* r\psi_b \mathrm{d}r = -\mathrm{e}\int \psi_b^* r\psi_a \mathrm{d}r. \tag{13}$$

It is easy to obtain equations for amplitudes a and b from (10) multiplying it by the corresponding eigenfunctions and integrating over the spatial variables

$$
\begin{aligned}
a_t + \mathrm{i}\omega_a a &= -\mathrm{i}\nu b \\
b_t + \mathrm{i}\omega_b b &= -\mathrm{i}\nu b.
\end{aligned} \tag{14}
$$

Here $\omega_a = E_a/\hbar$, $\omega_b = E_b/\hbar$ and $\nu = -p_0\mathcal{E}(x,t)/\hbar$.

For the atom moving with the velocity v we introduce internal and external reference frames connected by $x_e = x_i + vt$. Then electric field in the atom is given by

$$
\mathcal{E}(x_e, t) = E(x,t)\cos[k(x_i + vt) - \omega t + \phi(x,t)]. \tag{15}
$$

Difference in coordinates is negligibly small for the slowly varying amplitude and phase, but it contributes into the rapidly oscillating phase as Doppler shift $\Delta\omega = kv$. Let us make a substitution in (14)

$$
\begin{aligned}
a &= \mathrm{i}\alpha_1 \exp[-\mathrm{i}\omega_a(t - x_i/c) + \mathrm{i}t\Delta\omega/2] \\
b &= \alpha_2 \exp[-\mathrm{i}\omega_b(t - x_i/c) - \mathrm{i}t\Delta\omega/2]
\end{aligned} \tag{16}
$$

and suppose that intensity of the fields generated in the second harmonic $2\omega_0$ is small. We then obtain

$$
\begin{aligned}
\partial_t\alpha_1 + \frac{1}{2}\mathrm{i}\Delta\omega\alpha_1 &= \frac{p_0 E}{2\hbar}\mathrm{e}^{\mathrm{i}\phi}\alpha_2 \\
\partial_t\alpha_2 - \frac{1}{2}\mathrm{i}\Delta\omega\alpha_1 &= -\frac{p_0 E}{2\hbar}\mathrm{e}^{-\mathrm{i}\phi}\alpha_1.
\end{aligned} \tag{17}
$$

This is the so-called Zakharov-Shabat system, which appears in numerous problems to which inverse scattering method is applicable.

Normalized population density difference may be written as

$$
N = \frac{n}{n_0} = |\alpha_1|^2 - |\alpha_2|^2. \tag{18}
$$

Introducing $\Phi = kx_e - \omega t + \phi$ we can represent polarization p as

$$
p = p_0(P_1\cos\Phi + P_2\sin\Phi), \tag{19}
$$

where polarization envelopes P_1 and P_2 are given by

$$P_1 = \mathrm{i}(\alpha_1\alpha_2^*\mathrm{e}^{-\mathrm{i}\phi} - \alpha_2\alpha_1^*\mathrm{e}^{\mathrm{i}\phi}),$$
$$P_2 = -(\alpha_1\alpha_2^*\mathrm{e}^{-\mathrm{i}\phi} + \alpha_2\alpha_1^*\mathrm{e}^{\mathrm{i}\phi}). \tag{20}$$

As we can see, induced polarization has a component with amplitude p_0P_1, which oscillates in the same phase as electric field, but it also has a component oscillating with the phase shift $\pi/2$ and amplitude p_0P_2.

2.3 Model equations

Since there is a distribution of the atoms over different velocities, there exists corresponding distribution of the phase shifts $\Delta\omega$. Designating the latter as $g(\Delta\omega)$ and the total number of atoms in the unit volume as n_0 we can write the total polarization of the medium as

$$\mathcal{P}(x,t) = n_0\int\limits_{-\infty}^{\infty} g(\Delta\omega)p(\Delta\omega,x,t)\mathrm{d}\Delta\omega = n_0 < p(\Delta\omega,x,t) > . \tag{21}$$

Neglecting slow dependence on time of the envelopes P_1 and P_2 we shall obtain

$$\frac{\partial^2\mathcal{P}}{\partial t^2} \approx -\omega_0^2\mathcal{P} = -n_0p_0\omega_0^2 < (P_1\cos\Phi + P_2\sin\Phi) > . \tag{22}$$

Substituting (22,6) into (5) and neglecting second derivatives of slowly varying functions E and ϕ, as well as terms of the form $E_t\phi_t$, we can easily obtain that coefficients by the terms $\cos\Phi$ and $\sin\Phi$ satisfy respectively the following equations

$$\frac{\partial E}{\partial t} + c\frac{\partial E}{\partial x} = 2\pi n_0\omega_0p_0 < P_2 >,$$
$$E(\frac{\partial\phi}{\partial t} + c\frac{\partial\phi}{\partial x}) = 2\pi n_0\omega_0p_0 < P_1 > . \tag{23}$$

These equations may be written in the complex form as

$$[\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}](E\mathrm{e}^{\mathrm{i}\phi}) = 2\pi n_0\omega_0p_0 < P_2 + \mathrm{i}P_1 > \mathrm{e}^{\mathrm{i}\phi}. \tag{24}$$

To obtain equations for N, P_1 and P_2 let us differentiate (18) and (20)

$$\begin{aligned} \partial_t N &= -\frac{p_0 E}{\hbar} P_2, \\ \partial_t P_1 &= -(\Delta\omega + \phi_t) P_2, \\ \partial_t P_2 &= \frac{p_0 E}{\hbar} N + (\Delta\omega + \phi_t) P_1. \end{aligned} \tag{25}$$

System (25) is known as optical Bloch equations. Designating

$$P = (P_2 + \mathrm{i} P_1) \mathrm{e}^{\mathrm{i}\phi} \tag{26}$$

and changing notation by substituting E instead of $E\mathrm{e}^{\mathrm{i}\phi}$ we end up with the following system of equations

$$\begin{aligned} (\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})E &= 2\pi n_0 \omega_0 p_0 < P >, \\ \frac{\partial P}{\partial t} &= \frac{p_0 E}{\hbar} N - \mathrm{i}\Delta\omega P, \\ \frac{\partial N}{\partial t} &= -\frac{p_0}{2\hbar}(E^* P + P^* E). \end{aligned} \tag{27}$$

This system is called Maxwell-Bloch equations and serves as a mathematical model for purposes of the present work.

Let us shortly summarize physical meaning of variables in (27). Complex E describes the envelope of electromagnetic field in the medium. Complex P contains components of the polarization envelope, which oscillate in phase and with $\pi/2$ shift with respect to $\mathcal{E}$. N is normalized difference in population density between the two atomic levels, it varies between -1 and 1.

3 General properties of MB equations

We shall further consider rescaled and dimensionless system, which is more suitable for mathematical clarity

$$\begin{aligned} \frac{\partial E}{\partial t} + \frac{\partial E}{\partial x} &= < P >, \\ \frac{\partial P}{\partial t} &= EN - 2\mathrm{i}\Delta\omega P, \\ \frac{\partial N}{\partial t} &= -\frac{1}{2}(E^* P + P^* E), \end{aligned} \tag{28}$$

where as before brackets designate averaging with distribution function $g(\Delta\omega)$. Let us notice that the last two equations may be written in form similar to the Liouville equation for evolution of the density matrix $\hat{\rho}$ governed by the Hamiltonian $\hat{\mathcal{H}}$

$$\hat{\rho}_t = \mathrm{i}[-\Delta\omega\hat{\sigma}_z + \hat{\mathcal{H}},\, \hat{\rho}]$$

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\mathcal{H}} = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & E \\ -E^* & 0 \end{pmatrix}, \quad \hat{\rho} = \begin{pmatrix} N & P \\ P^* & -N \end{pmatrix}.$$

Most of the methods addressing this system are based on the inverse scattering method. Therefore we shall shortly describe it below. Let us consider the overdetermined system of equations for the matrix function ϕ, which we shall further call 'wavefunction'

$$\phi_t = \hat{U}(x.t, \lambda)\phi, \tag{30}$$

$$\phi_x = \hat{V}(x, t, \lambda)\phi, \tag{31}$$

where

$$\hat{U} = -\mathrm{i}\lambda\hat{\sigma}_z + \mathrm{i}\hat{\mathcal{H}} \tag{32}$$

$$\hat{V} = \mathrm{i}\lambda\hat{\sigma}_z - \mathrm{i}\hat{\mathcal{H}} + \frac{1}{4\mathrm{i}}\int_{-\infty}^{\infty} \frac{\hat{\rho}(\xi)g(\xi)}{\xi - \lambda}\mathrm{d}\xi. \tag{33}$$

The compatibility condition

$$\hat{U}_x - \hat{V}_t + [\hat{U}, \hat{V}] = 0 \tag{34}$$

is equivalent to the Maxwell-Bloch system (28).

Let us consider the class of rapidly decaying in t coefficients $E(x,t)$, meaning

$$\int\limits_{-\infty}^{\infty} |E(x,t)|\mathrm{d}t < \infty. \tag{35}$$

We introduce the scattering problem for (30) by defining the Jost functions $\chi^{\pm}$– solutions of (30) with given asymptotical behavior

$$\chi^{\pm} \to \exp(-\mathrm{i}\lambda\hat{\sigma}_z t), \quad t \to \pm\infty. \tag{36}$$

The scattering matrix S is defined by

$$\chi^{-} = \chi^{+} S,$$

and is of the form

$$S = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}.$$

Function $a(x,\lambda)$ is analytical in the upper half-plane $\Im\lambda > 0$ and has finite number of zeroes λ_j, provided (35) holds. These zeroes are eigenvalues of the spectral probem (30). Corresponding eigenfunctions are defined by their asymptotics

$$\psi_j \to \begin{pmatrix} 0 \\ \exp(\mathrm{i}\lambda_j t) \end{pmatrix}, \quad t \to \infty,$$

and then

$$\psi_j \to \begin{pmatrix} c_j \exp(\mathrm{i}\lambda_j t) \\ 0 \end{pmatrix}, \quad t \to -\infty.$$

Function $c(x,\lambda) = b^*(x,\lambda)/a(x,\lambda)$ and set of the constants λ_j and c_j constitute the scattering data.

Idea of the inverse scattering method is that the evolution of the scattering data in x turns out to be much simpler than that of E. This allows to separate the procedure of solution into three steps. First one has to obtain the scattering data given $E(t, x = 0)$. Then propagate the scattering data, which requires knowledge of the asymptotical behavior of P and N as $t \to \pm\infty$. And the last step is to use inverse scattering procedure to reconstruct E at any value of x. It involves solution of the system of linear integral equations, known as Gelfand-Levitan-Marchenko equations. We shall not describe IST method in more detail but rather point out few important details, which imply that a different method is required for a wide class of problems.

One of the most important limitations of IST method is the condition (35). The next obvious one is that it requires analysis of asymptotic behavior of the wavefunction, that is one has to consider the problem on the infinite interval. Most of the physical problems are however initial-boundary-value problems. For example we do not have knowledge about asymptotical behavior of polarization and population density in the medium, but rather know its inital state. The further analysis in this work is based on the ingenious observation by Krichever [6] that it is possible to solve initial-boundary-value problems for some nonlinear equations using Riemann-Hilbert problem.

4 Goursat problem

We study the following system

$$E_x = P,$$

$$P_t = EN, \tag{37}$$

$$N_t = -\frac{1}{2}(E^* P + EP^*).$$

It differs from the canonical Maxwell-Bloch equations (28) by the absence of the time derivative in the first equation. It corresponds to the reference frame moving with the speed of light, which is equal to 1 in dimensionless variables. The other important assumption is that $g(\Delta\omega) = \delta(\Delta\omega)$, i. e. we suppose that there is no Doppler shift in atomic frequencies. This reduction, however is not crucial and the simplified system still has very rich behavior. Hereafter we describe a method of treatment of this equation in a rectangular (or semi-infinite) domain $x \in [0 \ldots L]$, $t \in [0 \ldots T]$. The problem is shown to be well-posed if values of N and P are pescribed at $t = 0$ and E is set at $x = 0$. It is closely related to the fact that this system has two families of characteristics $x = \text{const}$ and $t = \text{const}$. The general method described below is not confined to MB equations and most of the technique may be extended to other nonlinear equations with separated in λ-plane poles of the Lax-pair.

Let us start from the nonlinear partial differential equation, or system of equations, in Lax representation

$$\hat{U}_x - \hat{V}_t + [\hat{U}, \hat{V}] = 0. \tag{38}$$

which arises as a compatibility condition of the overdetermined system of linear matrix equations

$$\begin{cases} \phi_t = \hat{U}(x.t, \lambda)\phi \\ \phi_x = \hat{V}(x, t, \lambda)\phi \end{cases} \tag{39}$$

It is known as a 'zero-curvature' condition $\phi_{tx} = \phi_{xt}$, yielding (38) in case of nondegenerate matrix function (wavefunction) ϕ. The parameter λ is called 'spectral parameter.' It plays crucial role in the solution of nonlinear equaion (38) via interplay between $x - t$ and λ dependences.

Although it is not necessary for some of the following derivations, we shall nevertheless specify certain symmetry properties and particular details of matrices $\hat{U}$ and $\hat{V}$, which allow us to carry out derivations specific to the Maxwell-Bloch equations.

$$\hat{U} = \mathrm{i}\lambda\hat{\sigma}_z + \mathrm{i}\hat{\mathcal{H}}(x,t) \tag{40}$$

$$\hat{V} = \frac{1}{4\mathrm{i}\lambda}\hat{\rho}(x,t). \tag{41}$$

Matrices $\hat{\mathcal{H}}$ and $\hat{\rho}$, introduced in the previous section are Hermitian with zero traces. This particular form of $\hat{U}$ and $\hat{V}$ yields that equation (38) is identical to the Maxwell-Bloch system (37). As a consequence the following relations hold

$$\hat{U}^{\dagger}(\lambda) = -\hat{U}(\lambda^*), \quad \hat{V}^{\dagger}(\lambda) = -\hat{V}(\lambda^*). \tag{42}$$

.

In the above expressions and further on symbols '$*$' and '$\dagger$' designate complex and Hermitian conjugation respectively. We shall hereafter assume that ϕ is a fundamental matrix of the system (39), meaning

$$\phi(0,0,\lambda) = \hat{I}, \tag{43}$$

which, considering symmetry properties of $\hat{U}$ and $\hat{V}$ yields that

$$\phi^{-1}(\lambda) = \phi^{\dagger}(\lambda^*), \quad \det\phi(x,t,\lambda) = 1. \tag{44}$$

The problem of our interest is as follows. Given the incoming pulse, which defines electric field E on the boundary of the active medium and initial state of the medium, recorded in the density matrix (P,N), we want to obtain value of the electric field at any point at any time inside of the medium. Mathematically we aim to solve the following initial-boundary-value problem. Given

$$\hat{\mathcal{H}}(0,t) = \hat{\mathcal{H}}_0(t), \quad \hat{\rho}(x,0) = \hat{\rho}_0(x), \tag{45}$$

$\hat{\mathcal{H}}(x,t)$ and $\hat{\rho}(x,t)$, evolving with respect to (38) and (40,41), are to be found. $\hat{\mathcal{H}}_0$ and $\hat{\rho}_0$ define $\hat{U}_0$ and $\hat{V}_0$ on the boundary and at the initial time respectively.

5 General solution

5.1 Wavefunction decomposition

¿From mathematical point of view treatment of the system (39) is complicated because of the two following reasons. First is that we have to solve 'two ordinary differential equations' simultaneously, both in x and t. The second reason is that matrices $\hat{U}$ and $\hat{V}$ are singular in complex λ-plane, i. e. $\hat{U}$ has a simple pole at infinity and $\hat{V}$ – at zero. This yields that ϕ has essential singularities at these points. To study behavior of the wavefunction in neighbourhoods of singular points let us perform the following decomposition.

Let us choose contour $\mathcal{L}$ in λ-plane, such that zero is inside of it and infinity is outside. We shall denote the inner region $\mathbf{N}_0$ and the outer region $\mathbf{N}_\infty$. Let us impose two regular Riemann-Hilbert (RH) problems. Find nondegenerate matrix functions G_0 and R_0 analytic in $\mathbf{N}_0$, G_∞ and R_∞ analytic in $\mathbf{N}_\infty$, such that the following relations hold on $\mathcal{L}$

$$R_\infty = G_0\phi, \tag{46}$$

$$R_0 = G_\infty\phi. \tag{47}$$

These problems will have the unique solution if we require that

$$R_\infty(x,t,\infty) = \hat{I}, \tag{48}$$

$$R_0(x,t,0) = \hat{I}. \tag{49}$$

We may extend solutions in regions, where they are not analytic using the same formulae (46,47). For additional details of RH-problems reader should refer to the Appendix.

Consider logarithmic time derivative of R_∞. It is analytic in $\mathbf{N}_\infty$; and in $\mathbf{N}_0$ is defined by

$$\partial_t R_\infty \cdot R_\infty^{-1} = \partial_t G_0 \cdot G_0^{-1} + G_0 \hat{U} G_0^{-1}. \tag{50}$$

This expression is clearly analytic in $\mathbf{N}_0$, since $\hat{U}$ is analytic in $\mathbf{N}_0$ and hence $\partial_t R_\infty \cdot R_\infty^{-1}$ is analytic everywhere, i. e. is a constant. Calculating it at infinity using (48) and taking the fact that R_∞ is nondegenerate into account, we conclude that $\partial_t R_\infty = 0$ and R_∞ is time-independent. This allows us to consider RH-problem (47) at $t = 0$.

Space evolution of the wavefunction ϕ is defined by $\hat{V}$ only and consequently $\phi(x, 0, \lambda)$ has the only singularity at zero. Therefore it solves RH-problem (46) with $G_0 = \hat{I}$. Since $\phi(0, x, \infty) = \hat{I} = R_\infty(x, \infty)$ and solution of regular RH-problem is unique when normalization is imposed, we conclude, that

$$R_\infty(x,\lambda) = \phi(0,x,\lambda). \tag{51}$$

In the same manner, considering logarithmic time derivative of R_0

$$\partial_x R_0 \cdot R_0^{-1} = \partial_x G_\infty \cdot G_\infty^{-1} + G_\infty \hat{V} G_\infty^{-1}. \tag{52}$$

we conclude that it is x-independent and RH-problem (47) maybe imposed for $x = 0$. As in the previous case we can easily see that because of the uniqueness of normalized solution of regular RH-problem

$$R_0(t,\lambda) = \phi^{-1}(0,t,0)\phi(0,t,\lambda). \tag{53}$$

Taking time derivative and using the fact that $(\phi^{-1})_t = -\phi^{-1}\phi_t\,\phi^{-1}$, we obtain

$$\partial_t R_0 = \mathrm{i}\lambda\phi^{-1}(0,t,0)\hat{\sigma}_z\phi(0,t,0)R_0. \tag{54}$$

Finally we can see that $R_0(t,\lambda)$ and $R_\infty(x,\lambda)$ maybe obtained from initial and boundary data using the following equations, which are the direct consequence of (51,54):

$$\partial_x R_\infty = \hat{V}_0 R_\infty, \tag{55}$$

$$\partial_t R_0 = \mathrm{i}\lambda\phi_0^{-1}\hat{\sigma}_z\phi_0 R_0, \tag{56}$$

$$\partial_t \phi_0 = \mathrm{i}\hat{\mathcal{H}}_0\phi_0 \tag{57}$$

with initial conditions

$$R_\infty(0,\lambda) = R_0(0,\lambda) = \phi_0(0) = \hat{I}. \tag{58}$$

To summarize the derivations above let us notice, that not only we separated singularities of the wavefunction ϕ in the λ-plane, but spatial and temporal variables as well. We obtained a nonlinear analogue of D'Alambert's formula for the wave equation expressing solution through initial and boundary values. In our case RH-problem serves as a connecting link to establish such superposition.

5.2 Wavefunction restoration

In the previous section it is described how one can obtain R_0 and R_∞ given initial and boundary data. The next step is to restore original wavefunction ϕ in order to obtain values of $\hat{U}$ and $\hat{V}$ at any x and t

$$\hat{U} = \phi_t \phi^{-1} \tag{59}$$

$$\hat{V} = \phi_x \phi^{-1}. \tag{60}$$

Let us introduce an auxillary wavefunction $\psi(x,t,\lambda)$ putting

$$\psi = R_\infty R_0^{-1}. \tag{61}$$

We can impose a regular RH-problem to find G_0 and G_∞ analytic in $\mathbf{N}_0$ and $\mathbf{N}_\infty$ respectively, connected on the contour $\mathcal{L}$ by

$$G_0 = \psi\, G_\infty. \tag{62}$$

Let us normalize G_∞ assuming

$$G_\infty(x,t,\infty) = \phi_0^{-1}(t). \tag{63}$$

Then we assign the following values to ϕ: inside $\mathbf{N}_0$

$$\phi = G_0^{-1} R_\infty \tag{64}$$

and inside $\mathbf{N}_\infty$

$$\phi = G_\infty^{-1} R_0. \tag{65}$$

Formulae for $\hat{U}$ and $\hat{V}$ follow: in $\mathbf{N}_0$

$$\hat{U} = \partial_t\, G_0^{-1} \cdot G_0, \tag{66}$$

$$\hat{V} = \partial_x\, G_0^{-1} \cdot G_0 + G_0^{-1} \hat{V}_0\, G_0 \tag{67}$$

and in $\mathbf{N}_\infty$

$$\hat{U} = \partial_t\, G_\infty^{-1} \cdot G_\infty + \mathrm{i}\lambda\, G_\infty^{-1}\, \phi_0^{-1}\, \hat{\sigma}_z\, \phi_0\, G_\infty \tag{68}$$

$$\hat{V} = \partial_x\, G_\infty^{-1} \cdot G_\infty. \tag{69}$$

G_∞ is analytic in $\mathbf{N}_\infty$ and, accounting (63), together with its inverse may be represented as

$$G_\infty = \phi_0^{-1}\,(\hat{I} + \frac{Q}{\lambda} + \ldots), \tag{70}$$

$$G_\infty^{-1} = (\hat{I} - \frac{Q}{\lambda} + \ldots)\,\phi_0. \tag{71}$$

Comparing analytic expansions of $\hat{U}$ and $\hat{V}$ in $\mathbf{N}_0$ and $\mathbf{N}_\infty$ and substituting equations (70,71) into (69,68) we obtain the final formulae

$$\hat{U} = \hat{U}_0 + \mathrm{i}[\hat{\sigma}_z, Q] \tag{72}$$

$$\hat{V} = -\frac{1}{\lambda} Q_x. \tag{73}$$

6 Quantum amplifier

Let us consider the following problem as an example of application of the above technique. Suppose the medium is initially excited and not polarized, i. e. $N = 1$, $P = 0$ at $t = 0$. We put a short pulse from one side of the medium and observe evolution of the system, expecting electric field to adsorb the energy from the medium as it propagates through.

$$\hat{V}_0 = \frac{1}{4\mathrm{i}\lambda}\hat{\rho}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\mathcal{H}}_0 = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & E_0(t) \\ -E_0^*(t) & 0 \end{pmatrix}. \tag{74}$$

Let us also suppose that incomimg pulse is short and carries little energy, i. e.

$$\int_0^\tau E_0(t)\mathrm{d}t = \epsilon \ll 1,$$

here τ is the duration of the pulse. We shall solve the problem in the linear approximation with respect to ϵ.

It is easy to solve (55), (56) and (57) to obtain

$$R_\infty = \begin{pmatrix} \mathrm{e}^{x/4\mathrm{i}\lambda} & 0 \\ 0 & \mathrm{e}^{-x/4\mathrm{i}\lambda} \end{pmatrix}, \quad R_0 = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\lambda t} & \epsilon \sin\lambda t \\ -\epsilon^* \sin\lambda t & \mathrm{e}^{-\mathrm{i}\lambda t} \end{pmatrix} + O(\epsilon^2) \tag{75}$$

for times $t \gg \tau$. Such structure of R_0 and R_∞ implies that one should seek solution of the RH-problem (62) in the form

$$G_\infty \approx \begin{pmatrix} e^{-x/4i\lambda} & \epsilon g_\infty \\ -\epsilon^* g_\infty^* & e^{x/4i\lambda} \end{pmatrix}, \quad G_0 \approx \begin{pmatrix} e^{-i\lambda t} & \epsilon g_0 \\ -\epsilon^* g_0^* & e^{i\lambda t} \end{pmatrix} \tag{76}$$

up to the terms quadratic in ϵ. This anzats allows us to reduce the homogenious matrix RH-problem (62) to the inhomogenious scalar one

$$g_0(x,t,\lambda) = e^{-i\lambda t} g_\infty(x,t,\lambda) + e^{x/2i\lambda} \sin \lambda t, \quad \lambda \in \mathcal{L}. \tag{77}$$

Solution of this problem is straightforward, though requires some technical calculations. Th e final answer is

$$E(x,t) \approx \epsilon x I_1(z)/z, \quad z = \sqrt{2tx}.$$

This formula describes the linear phase of development of the 'tail' of the wave front behind the passing pulse. Particulary interesting is self-similar dependence of E on z. It is worthy to mention that self-similarity appears in most of nonlinear soliton equations when solution describes some sort of free radiation. It is however obvious that this solution is not true for arbitrary large times. To extend it would require taking into account terms of higher order in ϵ and is out of the scope of this work.

7 Appendix

7.1 Riemann-Hilbert problem

The Riemann-Hilbert (RH) problem plays a major role in the theory of integrable systems. Let us notice however, that in some literature it is mentioned simply as Hilbert, or as Riemann problem. Here we call it paying tribute to the both authors.

Let us consider a closed non-selfintersecting contour $\mathcal{L}$ on the complex λ-plane and let a matrix function ψ be given on $\mathcal{L}$. We shall denote connected region 'outside' of the contour as $\mathbf{N}^-$ and the one 'inside' of $\mathcal{L}$ as $\mathbf{N}^+$. The problem is to find a sectionally holomorphic function G, analytic in $\mathbf{N}^+$ and $\mathbf{N}^-$, subject to the boundary condition

$$G^+ = \psi\, G^- \quad \text{on} \quad \mathcal{L}. \tag{78}$$

It is obvious that the solution is not unique. Indeed, if F is an arbitrary matrix function, constant in λ, then GF is also a solution of the stated problem. In order to fix the solution let us specify a normalization of the RH-problem by assigning to G a certain value at a certain point in λ-plane. In this work we only consider regular RH problem, when G is always nondegenerate, i. e. $\det G \neq 0$. Under these condition the solution is unique.

It is easy to reduce RH-problem to system of singular integral equations. Let us for simplicity fix normalaziation that $G^{-}(\lambda = \infty) = \phi$. Conditions of analyticity of G^{+} and G^{-} in corresponding regions yield

$$G^{+}(\zeta) = \frac{1}{\pi \mathrm{i}} \oint\limits_{\mathcal{L}} \frac{G^{+}(\xi)}{\xi - \zeta} \, \mathrm{d}\xi \tag{79}$$

$$G^{-}(\zeta) = 2\phi - \frac{1}{\pi \mathrm{i}} \oint\limits_{\mathcal{L}} \frac{G^{-}(\xi)}{\xi - \zeta} \, \mathrm{d}\xi. \tag{80}$$

Here $\zeta \in \mathcal{L}$ and integrals are considered in sense of principal value. Taking into account the jump condition (78) we can rewrite (79) as

$$G^{-}(\zeta) = \frac{\psi^{-1}(\zeta)}{\pi \mathrm{i}} \oint\limits_{\mathcal{L}} \frac{\psi(\xi) G^{-}(\xi)}{\xi - \zeta} \, \mathrm{d}\xi \tag{81}$$

Adding the last expression to (80) we obtain singular integral equation on $\mathcal{L}$ for G^{-}

$$G^{-}(\zeta) - \frac{1}{2\pi \mathrm{i}} \oint\limits_{\mathcal{L}} \frac{\psi^{-1}(\zeta)\psi(\xi) - \hat{I}}{\xi - \zeta} G^{-}(\xi) \, \mathrm{d}\xi = \phi. \tag{82}$$

Let us also notice, that the scalar RH-problem has an explicit solution in terms of Cauchy-type integrals

$$G(\zeta) = \phi \exp[\frac{1}{2\pi \mathrm{i}} \oint\limits_{\mathcal{L}} \frac{\ln \psi(\xi) \mathrm{d}\xi}{\xi - \zeta}]. \tag{83}$$

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