# Concept and computation: The role of curriculum*

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In *Knowing and Teaching Elementary Mathematics*, Liping Ma considers Chinese and U.S. teachers' understanding of topics in elementary mathematics.[1] In this article we will use some of the tools developed by Ma to discuss what might constitute such an understanding of a slightly more advanced topic: integers. This includes integer arithmetic and integers on the number line.

In talking about how to teach a particular topic, Ma and the Chinese teachers she interviewed discussed its connections with other topics. Such connections occurred both prior and later in the curriculum, and were made directly between topics and indirectly via general principles. The Chinese teachers discussed them as parts of "knowledge packages" for a given topic—conceptual and procedural topics that support and are supported by the learning of the topic. One teacher described it as a way of thinking in which one sees topics group-by-group rather than piece-by-piece:

> . . . you should see a knowledge "package" when you are teaching a piece of knowledge. And you should know the role of the present knowledge in that package. You have to know that the knowledge you are teaching is supported by which ideas or procedures, so your teaching is going to rely on, reinforce, and elaborate the learning of these ideas. (Ma, 1999, p. 18)

What might knowledge packages be for the integers? Obviously it depends on exactly how you delineate the topic and we'll see below that there are several extant variations. For a particular interpretation one can ask:

> What topics support the learning of this topic?
>
> What topics are supported by knowledge of this topic?

The answers to these questions constrain the placement of the topic in the curriculum. In Ma's book information about the Chinese national curriculum and textbooks occurs mainly to explain statements in the teachers' interviews. In contrast to Ma's focus on interviews with teachers, this article proceeds from examples of textbooks, curricula, and standards to questions about U.S. conceptions of a particular topic—the integers—and its

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[1]999 *MER Newsletter 12*(1), 1, 4–5, 10. Available at:
http://www.math.uic.edu/MER/pages/.

[1]All of the Chinese and almost all of the American teachers had experience teaching in only grades 1–6. (The first author mistakenly asserted in the May issue of the MER *Newsletter*. "The Chinese teachers were spread among grades K–8. . . ." The Chinese teachers in Ma's study were elementary teachers and "elementary school" in China is grades 1 to 6.)

connections with other topics, and more generally to suggest how such questions might be used in thinking about curricula.

*At what grades do negative numbers occur?*

Examples of U.S. textbooks, standards, and guidelines suggest that negative numbers have an uncertain relationship with the rest of the curriculum. Negative numbers may first occur in some form in any elementary grade. For example, an experimental curriculum of the 70s (CSMP) begins work with integer addition in 1st grade. The recently published *Math Trailblazers* discusses placement of marks on a line in 3rd grade and negative numbers on the number line in 4th grade. The reform textbook series *Mathland* introduces negative numbers in 6th grade, allocating a week to the negative numbers on the number line and their addition and subtraction (Unit 5, week 3, Integer Investigations). In contrast, the Algebra Project devotes a supplemental module spread over a semester of 6th grade to a thorough introduction of the number line and addition and subtraction of positive and negative numbers.[2]

The 1989 NCTM *Standards* urge the study in grades 5–8. Somewhat more ambitiously and precisely the recently adopted California Mathematics Standards lays out a progressive study of the negative numbers beginning in 4th grade. Negative numbers and the number line are introduced in the 4th grade, but arithmetic with negative numbers doesn’t occur in until 6th grade. According to these standards students should:

> “Use concepts of negative numbers (e.g., on a number line, in counting, in temperature, in ‘owing’)” in 4th grade.
>
> “Identify and represent on a number line decimals, fractions, mixed numbers, and positive and negative integers” in 5th grade.
>
> “Compare and order positive and negative fractions, decimals, and mixed numbers and place them on a number line” and “Add, subtract, multiply, and divide rational numbers (integers, fractions, and terminating decimals)” in 6th grade.

In contrast, the Japanese national curriculum introduces negative number topics much later. Although “number line” is mentioned in the list of terms and symbols for 3rd grade, it does not include negative numbers. The first mention of negative numbers occurs as part of the three objectives for the equivalent of 7th grade (grade 1 of lower secondary

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[2] A technical note may be in order here. One can't assume that mathematics lessons correspond directly with what's in the textbook. In Japan and China, however, teachers are expected to elaborate on what's in the textbook, but they do follow the order in which topics are presented (see e.g., Ma, 1999 or National Institute on Student Achievement, Curriculum, and Assessment, 1998). In the U.S., part of the role of the teacher is to choose topics from the textbook, perhaps with the assistance of state and district guidelines (Schmidt, McKnight, & Raizen, 1997).

school) and is further elaborated as one of the seven main mathematics topics for the grade:

> To enable students to understand the meaning of positive and negative numbers, and to compute with those numbers according to four fundamental operations. (Research Center for Science Education, 1989, p. 25)

The two approaches illustrated here (several short treatments vs. one long treatment) typify the results of the TIMSS curriculum study: On average, a given topic occurs more briefly and more often in U.S. textbooks and curricula than in those of other countries (Schmidt, McKnight, & Raizen, 1997).

*Where do negative numbers occur in textbooks?*

Can a gradual treatment such as that outlined by the California Standards prepare students for integer arithmetic? Or will such an approach suffer from the fragmentation discussed by Mayer, Sims, and Tajika (1995)? They studied lessons on the addition of signed numbers in three Japanese and four U.S. textbooks and found in "three U.S. books, material on addition and subtraction was in the same chapter as solving equations and coordinate graphing of equations; in another it was taught in a chapter that included units of measurement, mixed numbers, and improper fractions" (p. 456). A similar question concerns *Mathland*: in the 6th grade book, one week on integer addition and subtraction and the number line is preceded by a week on prime numbers and followed by a unit about areas of polygons.

In *Japanese Grade 7 Mathematics* the number line (with both positive and negative numbers), integer addition and subtraction, and integer multiplication and division occur in a chapter entitled "Positive and Negative Numbers." This is followed by "Letters and Expressions," a chapter about expressing situations in terms of variables and manipulating expressions that include variables.

*Meanings and models for negative numbers*

Part of the variability in occurrence of negative number topics in U.S. curricula may be due to different ways in which these topics are construed. Ball (1993) points out that negative numbers have at least two important aspects: they can be used to represent an amount of the opposite of something, and to represent a location relative to zero. Generalizing this slightly, the various models for negative numbers can be grouped into two categories: negatives represent an amount of the opposite of something (an additive inverse), or a location (direction and magnitude). CSMP, the curriculum used by Ball's school district illustrates negative numbers with a scenario involving magic and regular peanuts. If a magic and a regular peanut were in a pocket at the same time both disappeared. Concerned about fostering magical notions about mathematics, Ball rejected the peanut model for one involving paper people who went several floors up or down on a building with a ground floor of 0 (see Ball, 1993 for details). Part of her motivation for choosing this model was that it was a positional model like the number line, but might allow modeling of integer addition and subtraction (it turned out that Ball's third graders had difficulty interpreting 6 + (–6) in terms of elevator rides).

*Math Trailblazers* introduces Mr. O, the origin, in 3rd grade. In this short section, students mark places to the left and right and front of back of Mr. O, but negative numbers do not literally appear. There is a six-page section introducing negative numbers on the number line in grade 4; examples and problems include temperature, altitude, and bank balance. In grade 5, there is a full unit (2–3 weeks) devoted to maps and coordinates. This unit passes from concrete problems of maps to more abstract tranformations on the plane (what happens when an "L" is flipped across the *y*-axis?). Thus, these materials present a gradual introduction to the location aspect of negative numbers without any real connection to the arithmetic or "opposite" aspect.

In the grade 6 *Mathland*, the number line and integer addition and subtraction occur in the same week, but the two aspects of location and opposite do not seem connected in the activities or the text. The first day is devoted to location on the number line, the next three days to integer addition, and the last day to integer subtraction. Integer addition is introduced with pink cubes represent positive numbers and green cubes represent negative numbers. Pairs with both colors "cancel out so they equal 0." Students use this model for calculating sums of integers, and later for integer subtraction. According to the teacher's manual, the "key ideas" of this unit are: "Just as the number line extends infinitely in the positive direction, it also extends infinitely in the opposite, negative direction"; "Generalizations about the outcomes when integers are added or subtracted can be made based on relationships between the numbers" (p. 154). These are not connected with each other explicitly in the text and do not seem connected in the activities.

The 7th grade U.S. textbook examples discussed by Mayer, Sims, and Tajika suggest that connecting the two aspects of negative numbers may not have been an emphasis of all the textbook authors. Two of the models focused on opposite: a beaker containing positive and negative charges, and scenarios involving matter and antimatter. The other two models had elements of both opposite and location: Temperature and ratings (the ratings calculation was illustrated with a number line, suggesting a connection with location as well as opposite).

The models for integer addition presented in the three Japanese textbooks all illustrate opposite and location. However, location in two of the three models is not relative to zero, but to a given starting point. For example, *Japanese Grade 7* uses changes in water level in a tank. The initial point is "initial water level" on an unlabeled scale, a positive number corresponds to an increase in water level and a negative number corresponds with a decrease in water level. (Variations of this example are used later in the text to illustrate integer multiplication and division, e.g., if water flows in at 10 cm/min, how much higher is the water level 3 minutes later and how much lower was the water level 3 minutes earlier?) Another text uses the example of walking east or west from a given point A on an unlabeled road.

In devising the "Algebra Project" for grade 6, Robert Moses analyzed the role of positive and negative numbers in the differing meanings of subtraction for natural numbers and integers. The Algebra Project first introduces the notion of displacement. This is a deep treatment, motivated for students by a trip on public transit, and with an associated exploration of the notion of equivalence—vectors are equivalence classes. Then the students explore the difference between the old subtraction metaphor, "take away" and the new, "compare."

*Possible pieces of a knowledge package*

Moses's analysis suggests that a necessary piece of knowledge for learning negative numbers is the idea of subtraction as "compare" as opposed to "take away." First and second grade U.S. textbook word problems often focus on the "take away" interpretation of subtraction and the associated solution procedure of "sum – addend = [answer]" (Fuson, 1992). The associated strategy (known as "counting down") is often more difficult for students than strategies associated with different interpretations. A subtraction, say, 15 – 8 = ? can also be viewed as "How much *more* is 15 than 8?" which suggests the solution strategy of "counting up" from 8.[3]

The Algebra Project also suggests that students need to understand subtraction as the inverse of addition—a piece of knowledge that also occurs in the knowledge package for subtraction with regrouping (Ma, 1999, p. 19). Moses, Kamii, Swap, and Howard (1989) explain,

> Most [U.S.] algebra texts introduce subtraction as a transformed addition problem. Students are asked to think of subtraction (3 – (–2)) = +5) as "adding the opposite" or "finding the missing addend" (3 – ? = 5). . . . The problem is compounded because [U.S] students have overlearned "take-away" as the concept underlying subtraction. In algebra, "take-away" no longer has a straightforward application to subtraction. Within a couple of months of beginning algebra, students confront subtraction statements which have no discernable content, have only indirect meaning in relation to an associated addition problem, and are not at all obvious. (p. 434)

In 6th grade, one aspect of the Algebra Project is compensating for a mistaken emphasis in the first two years of school.

Is this focus necessary for learning about negative numbers—or only for learning algebra? Or do the two rely on some of the same pieces of knowledge? After working with students who had difficulty with algebra, Moses concluded that "the heart of the

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[3]Some elementary school projects and programs, for example the Cognitively Guided Instruction Project (Carpenter, Fennema, & Franke, 1996) and *Math Trailblazers* encourage many different addition and subtraction strategies. Teachers are aware of different possible strategies for solving the same problems and include problems associated with different interpretations of subtraction.

problem lay in their concept of number" (Moses et al., 1989, p. 432): In arithmetic students focus on magnitude, but in algebra students must be able to focus on magnitude and direction—"Which way?" as well as "How many?"

This suggests that negative numbers may be an important precursor for algebra, acting as what the Chinese teachers called a "concept knot"—a single concept that ties several important ones together (Ma, 1999, p. 78). Choosing models for the integers that tie magnitude and opposite together may be particularly important, for not only may they support students' coordination of both aspects of the integers, but may in turn support later learning of algebra.

Perhaps this is part of the reason the chapter on integers occurs in *Japanese Grade 7* immediately before the chapter on Letters and Expressions. Moreover, the two chapters have other themes in common: commutative, associative, and distributive laws; connections between lengths marked on a diagram and expressions involving numbers or symbols. The latter may support learning about functions and graphs (a later topic for grade 7 Japanese students).

The possible knowledge package sketched above is associated with a curricular approach involving one long treatment of the integers. However, the question of whether to take an in-depth or incremental approach seems less important than understanding which topics support and are supported by knowledge of the integers. Both teachers and curriculum writers need to consider such knowledge packages in determining where to place the study of various topics and how to establish connections among topics.

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