

Homework 12, Math 575A Fall 2008

1. Show that given a set of values f_j and a sequence of points $x_0 < x_1 < \dots < x_n$ in \mathbb{R} that

$$P(x) = P(x; x_j, f_j) = \sum_{j=0}^n f_j L_j(x)$$

where

$$L_j(x) = \frac{\prod_{k \neq j} x - x_k}{\prod_{k \neq j} x_j - x_k}$$

is the unique polynomial of degree $\leq n$ satisfying $P(x_j) = f_j$.

When $f_j = f(x_j)$ for some function $f(x)$, this is the *Lagrange formula* and when it is used to interpolate values of $f(x)$, we call this *Lagrange interpolation*

2. Recall that Rolle's theorem states that for any $f \in C^1([a, b])$ such that $f(a) = f(b) = 0$, there is a point $c \in (a, b)$ such that $f'(c) = 0$. Prove the following generalization:

For any $f \in C^n([a, b])$ such that $f(z_k) = 0$ for $a \leq z_0 < z_1 < \dots < z_n \leq b$, there is at least one point $c \in (a, b)$ such that $f^{(n)}(c) = 0$ (the n^{th} derivative vanishes at some interior point).

3. Let $f(x) \in C^{n+1}([a, b])$, and define $E(x) = f(x) - P(x; x_j, f(x_j))$. For all $t \in [a, b]$ with $t \neq x_k$. Apply your generalized Rolle's theorem to

$$G(x) = E(x) - \frac{\prod_{k=0}^n x - x_k}{\prod_{k=0}^n t - x_k} E(t)$$

bearing in mind that $G(x_k) = G(t) = 0$ to show that the $n + 1^{\text{st}}$ derivative of $G(x)$ vanishes at some interior point. Use that to show that for any $x \in [a, b]$, there is some $\xi \in (a, b)$ such that

$$f(x) - P(x; x_j, f(x_j)) = E(x) = \frac{\prod_{k=0}^n x - x_k}{(n + 1)!} f^{(n+1)}(\xi)$$

(Note: this result can be proved much more naturally by considering Newton's divided differences. In any event, this is the analogue of the Taylor remainder formula for Lagrange interpolation, and leads to many of the estimates that we will use later.)

4. Use Lagrange interpolation and your result from problem 3 to estimate the value the integral of $f(x)$ as a linear combination of the values $f_j = f(x_j)$,

$$\int_a^b f(x) dx \approx \sum_{j=0}^n c_j f_j.$$

Give explicit formulas for the coefficients c_j , and estimate the error in your formula in terms of a bound M on an appropriate derivative of $f(x)$. (Your answer will involve integrals of polynomial functions of x depending on the x_j . It is not necessary to evaluate these integrals, but, all unevaluated integrals should be independent of $f(x)$.)

5. Apply your results to derive the trapezoid rule and Simpson's rule with appropriate error bounds. These methods come from subdividing an interval into pieces on which you approximate the integrand as a polynomial (piecewise polynomial approximation). (Note: for the case of Simpson's rule, you will not get the best bound using this approach)
6. Implement a Lagrange interpolation routine and a numerical integration routine based on Lagrange interpolation in Matlab. Test them for exactness using polynomials of (appropriate) low degree, and for error scaling using $f(x) = e^x$ and $g(x) = \cos(x)$ on $[0, \pi]$. Your result should include listings and plots. Use both uniform spacing ($x_j = x/n$), and uneven spacing ($x_j = (j/n)^2$). Comment on where your methods agree with the predicted error scalings and where they do not. Note: Your implementation of Lagrange interpolation must work for arbitrary choices of the x_j .