CONJUGATE PTS in 2D HYDRODYNAMICS:

Basic idea: develop appr. Riem. Geom. tools to study fluids.

Quick recap:

- ARNOLD 1966: motions of ideal fluid in a fixed domain M (compact, Riem.) $\Longleftrightarrow$ geodesics of KE metric (ess. L^2-inner product) in $D_\mu(M)$ (μ - Riem. vol. form).

- EBIN-MARSDEN 1970: well-def. C^∞ Riem. exp map on $D^s_\mu(M)$ (Sobolev H^s, $s > \frac{\dim M}{2} + 1$):

$$\exp_e : T_e D^s_\mu \to D^s_\mu$$

$$T_e D^s_\mu = H^s(TM)_{\text{div-free}}$$

defined by: $\exp_e t u_0 := \gamma(t)$

unique geodesic of KE in D^s_μ with $\gamma(0) = e$, $\dot\gamma(0) = u_0$.

$\implies$ • $\exp_e$ is „Lagrangian" solution map of:

$$\begin{cases} u_t + \nabla_u u = -\operatorname{grad} p \\ \operatorname{div} u = 0 \\ u(0) = u_0 \end{cases}$$

• $\exp_e$ is a local diffeo

so locally O.K.

„beyond local"

$\implies$ Singularities of $\exp_e$ are the conjugate pts.

More precisely:

$\eta = \exp_e t_c u_0$ is <u>conjugate to e</u> along $\gamma(t) = \exp_e t u_0$

if $d\exp_e(t_c u_0)$ is singular.

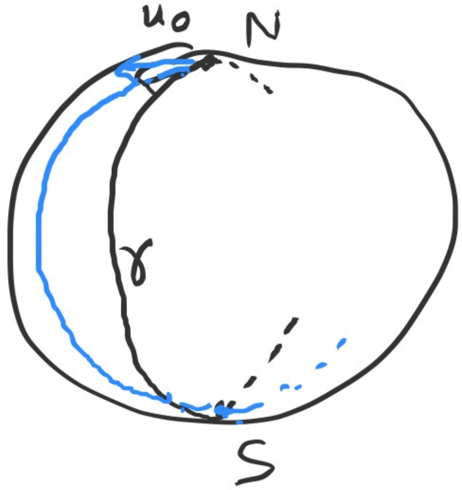
or if $\exists$ non-zero <u>Jacobi field</u> $J(t)$ with $J(0) = 0 = J(t_c)$ on $\gamma(t)$

$J''(t) + R^{D\mu}(J(t), \dot\gamma)\dot\gamma = 0.$

• Examples:

• Great circles on $S^2 \subset \mathbb{R}^3$ with round metric

(classical R.G.)

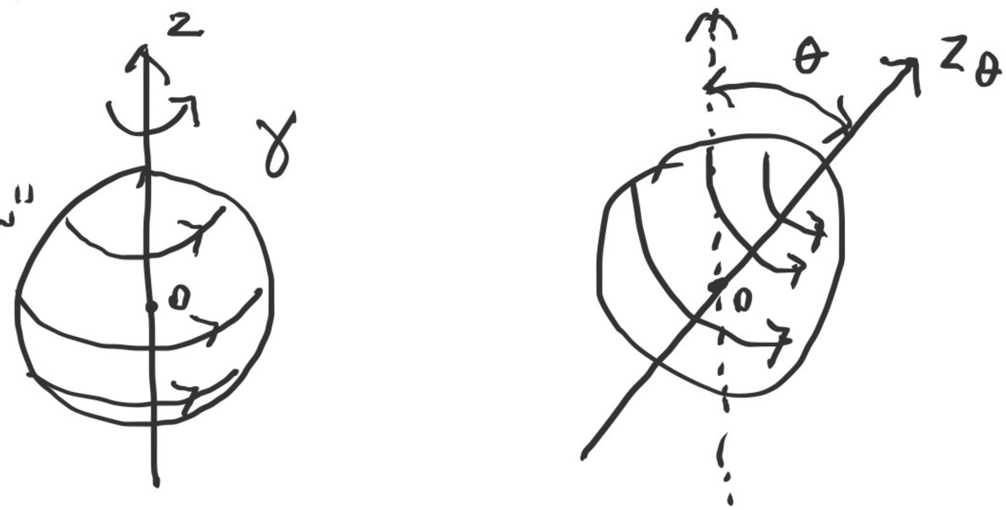


N, S are conjugate along any great circle

(Fluids)

• Rigid rotations of S^2 in $\mathcal{D}^s_\mu(S^2)$:

rotational "zonal flow" around z-axis



$\Longrightarrow$ $\gamma(0)$ and $\gamma(\pi)$ are conjugate along γ in $\mathcal{D}^s_\mu(S^2)$

(0). Explain meaning and role of conjugate pts in fluid dynamics!

(1). Construct explicit examples of conj. pts along non-stationary geo's in $\mathcal{D}^s_\mu(M^2)$

(2). Do all Kolmogorov flows on $\mathbb{T}^2$ which which are not uni-directional possess conjugate pts?

$\cos(kx+ly)$

$k, l \in \mathbb{Z}$

(e.g. $k=l$?)

(3). Determine the order of conjugacy of the first conj. pt along any geodesic in $\mathcal{D}^s_\mu(M^2)$ starting from e.

- Can it be $=1$?
- Is it always >1?

(4). Determine whether any two diffeo's in $\mathcal{D}^s_\mu(M^2)$ can always be connected by a minimizing geodesic?

(5). What is the relation between existence of conj. pts in $\mathcal{D}^s_\mu(M^2)$ and Arnold stability of stationary flows in M^2?

$$\forall\, v \in T_e\mathcal{D}^s_\mu, \quad \|v\|_{L^2} = 1$$

$$m_c^{u_0,v} := \langle R^{\mathcal{D}^s_\mu}(u_0,v)\rangle_{L^2} - \|P_e \nabla_{u_0} v\|^2_{L^2}$$

$$m_c^{u_0,v} = -2\underline{\underline{E''_{u_0}(v)}} + \langle [\mathrm{ad}_v, \mathrm{ad}^*_v] u_0, u_0\rangle_{L^2}.$$

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