Local normal forms without group action

On local classification of higher order linear ordinary differential operators at a singular point

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Hamiltonian Systems seminar at UoT, March 2, 2021

What means "classification"?

Let $\mathscr{X}$ a set (space, variety, *universum*, finite- or infinite-dimensional) of objects and G a (Lie) group acting on $\mathscr{X}$:

$$G \times \mathscr{X} \to \mathscr{X}, \quad (g, X) \mapsto g \cdot X. \qquad \operatorname{Orbit}(X) = \{g \cdot X \colon \forall g \in G\}.$$

Classification: description of orbits of this action, starting from the "largest" orbits and down to certain codimension (ignoring highly degenerated small orbits). Orbits are labeled by a "simplest" representative, normal form.

Example (Matrix conjugacy)

$$\mathscr{X} = \operatorname{Mat}(n, \mathbb{C}), \quad G = \operatorname{GL}(n, \mathbb{C}), \quad H \cdot X = HXH^{-1}.$$

Biggest orbits: diagonal matrices. Degenerate cases (multiple eigenvalues): Jordan normal forms with different block structures.

Infinite-dimensional examples

Example (Germs of vector fields,—analytic, smooth, formal)

$$\mathscr{X} = \{v\colon (\mathbb{R}^n, 0) \to \mathbb{R}^n = 0\}, \quad G = \{H\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \text{ invertible}\},$$
$$H \cdot v = H_*^{-1}(v \circ H), \quad H_* = \text{Jacobian (linearization) of } H \text{ at } 0.$$

Biggest orbit: nonsingular, $v = \text{const} \neq 0$.
Largest orbits in the stratum $\{v(0) = 0\}$: linear fields $v(x) = \Lambda x$, Λ diagonal.
Degenerate cases: resonances (identities between eigenvalues of $A = v_*(0)$).

Example (Linear first order systems)

$$\mathscr{X} = \mathscr{A}_n = \{A : \dot{x} = A(t)x,\ x \in \mathbb{C}^n\}, \quad G = \{H\colon t \mapsto H(t) \in \mathrm{GL}(n, \mathbb{C})\},$$
$$H \cdot A = B, \qquad B(t) = \dot{H}(t)\,H^{-1}(t) + H(t)\,A(t)\,H^{-1}(t).$$

Action: **gauge transform**, change of variables in the system $\dot{x} = A(t)x$ by $x = H(t)y$ for $x, y \in \mathbb{C}^n$. Various flavors: global $t \in \mathbb{C}P^1$, local $t \in (\mathbb{C}, 0)$.

Continuation: **singularities** of (formal) linear systems

$\mathbb{C}[\![t]\!]$ the ring of *formal power series* in t, $\Bbbk = \mathbb{C}[\![t]\!](t)$ the field of fractions (Laurent formal series with finitely many negative powers).

Let $A \in \operatorname{Mat}(n, \Bbbk)$ a formal *matrix* Laurent series:

$$A(t) = t^{-r}A_{-r} + \cdots + A_0 + A_1 t + \cdots + A_n t^n + \cdots, \quad r \in \mathbb{N} < +\infty.$$

We consider "systems of differential equations" of the following form:

$$\epsilon x = A(t)x, \qquad \epsilon = t\,\frac{\mathrm{d}}{\mathrm{d}t} \quad \text{the Euler derivation, } \epsilon x = t\dot{x}.$$

Why ϵx and not just $\dot{x} = \frac{\mathrm{d}}{\mathrm{d}t}x$? Algebraic "convenience". The usual derivative is nilpotent on nonnegative monomials t^n, while ϵ is diagonal.

The Lie group $G = \mathrm{GL}(n, \mathbb{C}[\![t]\!])$. Action: *formal gauge equivalence*.

Pre-classification: ($r \geqslant 0$ the *Poincaré index*).

1. $A \in \operatorname{Mat}(n, \mathbb{C}[\![t]\!])$: Fuchsian case, $r = 0$. The leading term is A_0. (If $A(0) = A_0 = 0$, we have a *nonsingular* case in the ODE sense.)
2. A has negative powers (non-Fuchsian, $r > 0$, irregular case).

Main theorem in the simplest case (Poincaré, . . . , end'XIX)

Theorem (Classification of nonsingular and Fuchsian systems)

A Fuchsian system from $\mathscr{A}_n$

$$\epsilon x = A(t)x, \qquad A(t) = A_0 + A_1 t + A_2 t^2 + \cdots \in \operatorname{Mat}(n, \mathbb{C}[\![t]\!]),$$

is formally gauge equivalent to its truncation at the constant leading term

$$\epsilon x = A_0 x$$

if no two eigenvalues $\lambda_1, \dots, \lambda_n$ of the leading matrix A_0 differ by a nonzero natural number: $\lambda_i - \lambda_j \notin \mathbb{N}$ (the non-resonance condition).

Note that this non-resonance condition holds automatically if $A_0 = 0$ (in the non-singular case). When the series for $A(t)$ is convergent, the gauge transform $H(t)$ can also be chosen convergent.

Degenerate (resonant) systems can be classified completely: one will have to keep some higher order terms (very sparse matrices A'_n for $n > 0$), but the *normal form* $A'(t) = A_0 + A'_1 t + \cdots + A'_n t^n + \cdots$ will still be integrable.

Are there other linear ordinary differential equations?

But of course! There are *higher order* linear differential equations of the form

$$a_0(t)\,y^{(n)} + a_1(t)\,y^{(n-1)} + \cdots + a_{n-1}(t)\,y' + a_n(t)\,y = b(t), \quad a_k, b \in \Bbbk,$$

where $y^{(k)} = \frac{\mathrm{d}^k}{\mathrm{d}t^k}y$ stands for the kth derivative of the unknown function y in t.

It is assumed that $a_0(t) \not\equiv 0$ (otherwise the order n of the equation can be reduced). The equation is homogeneous, if $b(t) \equiv 0$. As before, we can consider various settings regarding the coefficients a_k if they are non-constant: they may be germs at the origin (singularity), formal Laurent series or fully fledged functions with non-local domains.

Note that in the homogeneous case $b \equiv 0$ one can multiply both sides by any finite power t^n, eliminating poles of coefficients.

An equation can be reduced to a system of first order for variables $x_1 = y,\ x_2 = y',\ \dots,\ x_n = y^{(n-1)}$ with only the last line "individual". Such structure is called a *companion form*.

Conversely, any coordinate x_k of $\dot x = A(t)x$ satisfies an equation of order n.

The kth rows of consecutive derivations of the initial sytem, $\frac{\mathrm{d}^i}{\mathrm{d}t^i}x = A_i(t)x, \quad A_{i+1} = \dot A_i + A_iA_0, A_0 = A(t)$, will eventually become linear dependent over $\Bbbk$ no later after n steps. This translates into a differential identity between the derivatives $x_k^{(i)}$.

One theory or two?

From now on we will deal with homogeneous equations only, of the form $Ly = 0$, where L is a differential operator.

Do we have a gauge group action?—No!

Consider an equation $Ly = 0$ of order n, reduce it to a companion system $\dot{x} = Ax$ from $\mathscr{A}_n$. But after a gauge transform the system loses its companion form! Reduction back to a scalar higher order equation is problematic.

The natural "gauge transforms" $y = h(t)u$ preserving the companion structure form a group obviously too small to have interesting orbits.

Can we "doctor" the idea of "changing of variables" to mimic action of groups $G = \mathrm{GL}(n, \cdot)$ on the higher order equations? Looks like this is possible.

$$u = h_0(t)y + h_1(t)y' + \cdots + h_{n-1}(t)y^{(n-1)} + \cdots$$

The series may be truncated (removing all higher derivatives in . . .) if we are going to apply it to nth order equations.

- Can one organize such "transformations" into a group? in particular,
- How this change of variables should be inverted? Do we need to solve differential equations? what about the initial conditions?

Differential operators and Ø. Ore (1932–1933) calculus

Differential equations can be described using differential operators or "noncommutative univariate polynomials" $L \in \Bbbk[z]$ with coefficients from a field $\Bbbk$ and a *noncommutative variable* z. (Think of $z = \partial = \frac{\mathrm{d}}{\mathrm{d}t}$ or $z = \epsilon$.)

The key point is the commutation law. We consider only the rules which *are compatible with the notion of degree in* z:

$$az - za = b, \qquad a, b \in \Bbbk. \tag{1}$$

It *implies* the Leibnitz rule, e.g., when $\Bbbk$ is a differential field and z is a (first order) differential operator. We denote $b = a'$ to support this mnemonics.

Any $L \in \Bbbk[z]$ can be uniquely presented as $L = \sum_{i=0}^{n} a_i z^i$, $a_i \in \Bbbk$ all to the left from z^i. Computing products needs using the rule (1).

Theorem (Euclid algorithm)

For any $F, G \in \Bbbk[z]$ *with* $\deg G \leqslant \deg F$ *there exist*

$$F \overset{\text{right}}{=} GQ + R \overset{\text{left}}{=} PG + S.$$

Corollary: $\text{r-lcf}(A, B) = M$ *is well defined:* $M = HA = GB$.

Factorization and homogeneous linear ODE's, I

Let $z = \partial$ or $z = \epsilon$ and $\Bbbk$ a differential field (e.g., meromorphic germs). Then $Ly = 0$ is a *differential equation* (solutions usually are outside of $\Bbbk$).

Divisibility $F = QG$ means that $\{Gy = 0\} \subseteq \{Fy = 0\}$. All the way around, if $\{Gy = 0\} \subseteq \{Fy = 0\}$, then $F = QG$ (G is a right divisor).

What does the change of variables $u = h_0(t)y + h_1(t)y' + \cdots + h_{n-1}(t)y^{(n-1)} + \cdots$? Replaces y by $u = Hy$.

Equation: $Ly = 0$. **Change of variables**: $u = Hy$. **How to transform** L?

How to write an equation $Mu = 0$ for u?

$$0 = Mu = MHy \quad \forall\{y : Ly = 0\} \iff MH \text{ divisible by } L \iff MH = GL.$$

Pre-definition

Two linear operators $L, M \in \Bbbk[z]$ are "conjugated" by an operator $H \in \Bbbk[z]$, if there exists operator $G \in \Bbbk[z]$ such that $MH = GL$ in $\Bbbk[z]$.

Factorization and homogeneous linear ODE's, II

Pre-definition

Two linear operators $L, M \in \Bbbk[z]$ are "conjugated" in $\Bbbk[z]$ if there exist operators $G, H \in \Bbbk[z]$ such that $MH = GL$.

We need to exclude the trivial case $H = L$, $G = M$.

The change of variables H must be *faithful*: no solution of $Ly = 0$ is sent to zero by H. Algebraically, $\{Ly = 0\} \cap \{Hy = 0\} = 0$, that is,

$$\text{r-gcd}(H, L) = 1.$$

Definition (Ore conjugacy, O-conjugacy)

Two linear operators $L, M \in \Bbbk[z]$ are O-conjugated if there exists $G, H \in \Bbbk[z]$ such that $MH = GL$ and $\text{r-gcd}(L, H) = 1$.

Questions.

1. Is O-conjugacy a genuine equivalence relation (symmetric, reflexive, transitive)?
2. Is it induced by some clandestine group action?

Answers.

1. Yes, although the reflexivity (symmetry) is not at all obvious.
2. Doesn't seem likely, although I don't know any proof.

Field of fractions $\Bbbk = \mathbb{C}[\![t]\!](t)$ vs. the ring $\mathbb{C}[\![t]\!]$.

The local case: $\Bbbk = \mathbb{C}[\![t]\!](t)$ is the field of fractions (Laurent series) of the ring $\mathbb{C}[\![t]\!]$ of formal Taylor series and $z = \epsilon$.

Why not $z = \partial_t$? Any operator $L \in \Bbbk[z]$ can be multiplied *from the left* by a suitable t^k, $k \in \mathbb{Z}$, so that it falls into $\mathbb{C}[\![t]\!][z]$ (cancellation of poles).

This multiplication does not affect the homogeneous equation, $Ly = 0$. When studying homogeneous differential equations, we can always assume that their coefficients are from $\mathbb{C}[\![t]\!]$.

Obviously, the algebra $\Bbbk[z]$ is the same for $z = \partial_t$ and $z = t\partial_t$ (re-expansion).

However, the algebras $\mathbb{C}[\![t]\!][\partial]$ and $\mathbb{C}[\![t]\!][\epsilon]$ are different. Pay attention!

Remark. Working with Taylor series (even formal) is much more convenient: you always know where the leading term of any expansion is. Avoiding division in $\mathbb{C}[\![t]\!]$ is relatively easy.

Weyl-type algebra $\mathscr{W}$: some anatomy

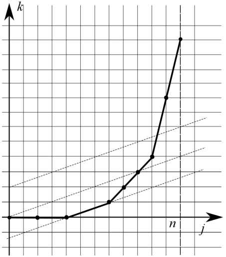
$\mathscr{W} = \mathbb{C}[\![t]\!][\epsilon]$ the **Weyl-type algebra**[1] (only nonnegative powers of t):

$$\mathscr{W}_n = \bigl\{L = \textstyle\sum_{k\geqslant 0} t^k p_k(\epsilon) : \ p_k \in \mathbb{C}[\epsilon], \deg p \leqslant n\bigr\}, \qquad \mathscr{W} = \textstyle\bigcup_n \mathscr{W}_n.$$

The commutation law in (t, ϵ)-variables reads $\ p(\epsilon)\, t^k = t^k\, p(\epsilon + k)$.

The commutation law would look much less simple if applied to $z = \partial$: $p(\partial)t^k$ will be a sum of many terms $t^i q_i(\partial)$ with $i \leqslant k$.

Any $L \in \mathscr{W}$ can be expanded in the noncommutative series



$$L = \sum_{k\geqslant 0}\sum_{j=0}^{n} c_{jk} t^k \epsilon^j, \qquad c_{jk} \in \mathbb{C}.$$

Support: $\operatorname{supp} L = \{(j,k) : c_{jk} \neq 0\} \subset [0\,..\,n] \times \mathbb{Z}_+ \subseteq \mathbb{Z}^2$.

Newton polygon:

$$\Delta_L = \operatorname{conv}\bigl(\operatorname{supp} L \overset{\text{Minkowski}}{+} \mathbb{Z}'_+\bigr), \qquad \mathbb{Z}'_+ = \{0\} \times \mathbb{Z}_+ \subseteq \mathbb{Z}_2.$$

[1] The classical Weyl algebra is $\mathbb{C}[t, \partial]$ with the Leibniz rule $[\partial, t] = 1$.

Fuchsian operators

Definition (Fuchsian operators, class $\mathscr{F}_n$)

An operator $L = \sum_{k \geqslant 0} t^k p_k(\epsilon) \in \mathbb{C}[\![t]\!][\epsilon] = \mathscr{W}$ of order n is called Fuchsian, if $\forall k \geqslant 0 \ \deg_\epsilon p_k \leqslant n$ and $\deg_\epsilon p_0 = n$.

Raison d'être: specific properties of Fuchsian equations $Ly = 0$, $L \in \mathscr{F}$.

Correspond to: Fuchsian systems $\epsilon x = Ax$, $A \in \operatorname{Mat}(n, \mathbb{C}[\![t]\!])$.

Newton diagram: vertical semistrip $[0, n] \times \mathbb{R}_+ \subseteq \mathbb{R}^2$.

Remark. $\mathscr{F} = \bigcup_{n \geqslant 0} \mathscr{F}_n$ is closed by composition in $\mathscr{W}$ but is not a subalgebra (leading terms may cancel each other after summation).

Definition (Non-resonant Fuchsian operators)

$L \in \mathscr{F}$ is non-resonant, if no two roots λ_i, λ_j of the characteristic polynomial $p_0(\lambda)$ differ by a natural $m \in \mathbb{N}$: $\lambda_i - \lambda_j \notin \mathbb{N}$.

Equivalent definition: $\forall m \in \mathbb{N} \quad \gcd\big(p_0(\lambda), p_0(\lambda + m)\big) = 1.$

Flavors of O-equivalence on $\mathscr{W}$

Definition (O-conjugacy in $\mathscr{W}$-flavor)

Two linear operators $L, M \in \mathscr{W}$ are W-conjugated[a] if there exists $G, H \in \mathscr{W}$ such that $MH = GL$ and $\text{r-gcd}(L, H) = 1$.

[a] Originally the O-conjugacy was for $\Bbbk[z] = \mathbb{C}[\![t]\!](t)[\epsilon]$ and not for $\mathscr{W} = \mathbb{C}[\![t]\!][\epsilon]$.

1. Which operators L we want to classify up to W-conjugacy? Nonsingular, Fuchsian, arbitrary irregular (non-Fuchsian)?—Simplest first.
2. Which operators H, G we allow for the W-conjugacy?
   1. Nonsingular (up to a multiple of t^k)
   2. Singular Fuchsian: $F, G \in \mathscr{F} \subsetneq \mathscr{W}$
   3. Arbitrary (irregular) from $\mathscr{W}$

The choice is nontrivial: the corresponding equivalence will be coarser or finer, cf. with holomorphic/meromorphic classification.

Experimental result (Shira Tanny, S.Y.) + **Definition**

Only the choice of Fuchsian operators $H, G \in \mathscr{F} \subsetneq \mathscr{W}$ for W-conjugacy leads to a nontrivial classification. We call it F-equivalence on $\mathscr{W}$.

Main theorem for Fuchsian systems (recall)

Theorem (Classification of Fuchsian systems)

A Fuchsian system from $\mathscr{A}_n$

$$\epsilon x = A(t)x, \qquad A = A_0 + A_1 t + A_2 t^2 + \cdots \in \operatorname{Mat}(n, \mathbb{C}[\![t]\!]),$$

is formally gauge equivalent to its truncation at the leading term

$$\epsilon x = A_0 x$$

if no two eigenvalues $\lambda_1, \ldots, \lambda_n$ *of the leading matrix* A_0 *differ by a nonzero natural number:* $\lambda_i - \lambda_j \notin \mathbb{N}$ (*the non-resonance* condition).

Degenerate (resonant) systems can be classified completely: one has to keep some higher order terms (very sparse matrices A'_n for $n > 0$), but the *normal form* $A'(t) = A_0 + A'_1 t + \cdots + A'_n t^n + \cdots$ will still be integrable.

Main theorem for systems in the simplest case

Theorem (F-Classification of Fuchsian operators, Shira Tanny and S. Y., *Arnold Math J.*, 2015)

An operator from $\mathscr{F}_n$

$$L = p_0(\epsilon) + tp_1(\epsilon) + \cdots + t^k p_k(\epsilon) + \cdots \in \mathbb{C}[\![t]\!][\epsilon]$$

is formally F-equivalent to its truncation at the leading term

$$L = p_0(\epsilon),$$

if no two roots λ_i, λ_j *of the leading term* p_0 *differ by a natural number,* $\lambda_i - \lambda_j \notin \mathbb{N}$ (*the non-resonance condition*).

Degenerate (resonant) equations can be classified completely: one has to keep some higher order terms (very sparse polynomials $q_k \in \mathbb{C}[\epsilon]$ for $k > 0$ in the normal form $M = \sum_{k \geqslant 0} t^k q_k$), but the normal form will be integrable.

Strange twin algebras

The proof is achieved by calculations in the Weyl algebra $\mathscr{W} = \mathbb{C}[\![t]\!][\epsilon]$, which are surprisingly similar to calculations in the Lie group $\mathrm{GL}(n, \mathbb{C}[\![t]\!])$.

Both are formal series $\sum t^k p_k(\epsilon)$, $p_k \in \mathbb{C}[\epsilon]$, resp., $\sum t^k A_k$, $A_k \in \mathrm{GL}(n, \mathbb{C})$.

In the first case the non-commutativity is *between the variables* t, ϵ, while polynomials $\mathbb{C}[\epsilon]$ form a commutative ring. In the second case t commutes with matrices, but the group $\mathrm{GL}(n, \mathbb{C})$ *itself is non-commutative*.

In the (classical Poincaré) proof elimination of all non-principal terms rests upon solving (with respect to H) the matrix *homological equation*

$$[A_0, H] + nE = R, \qquad A_0 \text{ the leading matrix, } n \in \mathbb{N}, \ R \text{ any r.h.s.}$$

In the Weyl case one has to solve w.r.t. $u, v \in \mathbb{C}[\epsilon]$ the *Sylvester equation*

$$u p_0 + v p_0^{[n]} = w, \qquad p_0 \in \mathbb{C}[\epsilon], \quad p_0^{[n]}(\,\cdot\,) = p_0(\,\cdot\, + n).$$

Any higher reasons for this parallelism, anybody?

F-equivalence, revisited

Definition (F-equivalence), recall.

Two linear operators $L, M \in \mathscr{W}$ are F-conjugated if there exist $G, H \in \mathscr{F}$ such that $MH = GL$ and $\text{r-gcd}(L, H) = 1$.

To "transform" M by H (or L by G), we need to find another factorization of the noncommutative product MH so that the term L, "similar" to M, would appear to the right.

What means this "similarity"? What do we know about noncommutative factorization in general and in the specific case of $\mathscr{W}$ in particular?

The general theory of noncommutative factorization in $\Bbbk[z]$ was developed by ØYSTEIN ORE in the same papers (1932–1933).

In the specific case of $\mathscr{W} = \mathbb{C}[\![t]\!][\epsilon]$ we have much more precise description in terms of the Newton polygon.

Weyl-type algebra $\mathscr{W}$: Newton polygon (recall)

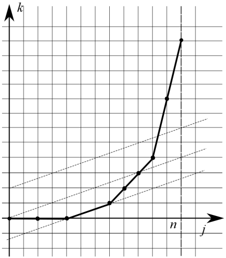
$\mathscr{W} = \mathbb{C}[\![t]\!][\epsilon]$ the **Weyl-type algebra** (only nonnegative powers of t):

$$\mathscr{W}_n = \left\{L = \sum_{k\geqslant 0} t^k p_k(\epsilon) : \; p_k \in \mathbb{C}[\epsilon], \deg p \leqslant n\right\}, \quad \mathscr{W} = \bigcup_n \mathscr{W}_n.$$

The commutation law: $p(\epsilon)\, t^k = t^k\, p(\epsilon + k)$.

Series expansion: t^k to the left from ϵ^j (canonical form),

$$L = \sum_{k\geqslant 0} \sum_{j=0}^{n} c_{jk} t^k \epsilon^j, \qquad c_{jk} \in \mathbb{C}.$$



Support: $\operatorname{supp} L = \{(j,k) \colon c_{jk} \neq 0\} \subset [0\,..\,n] \times \mathbb{Z}_+ \subseteq \mathbb{Z}^2$.

Newton polygon:

$$\Delta_L = \operatorname{conv}\bigl(\operatorname{supp} L \overset{\text{Minkowski}}{+} \mathbb{Z}'_+\bigr), \quad \mathbb{Z}'_+ = \{0\} \times \mathbb{Z}_+ \subseteq \mathbb{Z}_2.$$

Mnemonics: t “small”, ϵ “large” in the sense of domination between monomials. In particular, $t^k \epsilon^j = \epsilon^j t^k + \cdots$ (dominated terms).

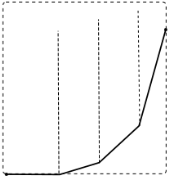
Factorization and Newton polygons, I

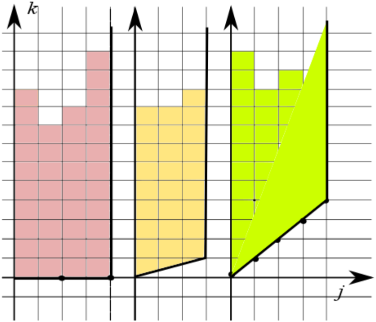
$\Delta\colon \mathscr{W} \ni L \longmapsto \Delta_L \subseteq \mathbb{R}^2$ the Newton polygon.

The **logarithm rule**:

$$\Delta_{LM} = \Delta_L + \Delta_M = \Delta_{ML}$$

($+$: Minkowski sum).





Proposition.
Any (admissible) Δ can be uniquely expanded as $\Delta = \Delta_1 + \cdots + \Delta_\nu$, Δ_i single-slope polygons.

Admissibility:
(1) vertices only at the lattice points, (2) lower left corner at the origin, (3) invariant by upwards shift.

Remark. If the lower edge of a single-slope polygon contains no internal lattice points, then Δ_i is irreducible in the class of admissible polygons, otherwise one can subdivide (expand) it further.

Miracle. Non-commutativity of $\mathscr{W}$ is not an obstacle to the logarithm rule.

Factorization and Newton polygons, II

Theorem (Factorization of operators from $\mathscr{W}$)

Any expansion of Δ_L into single-slope polygons Δ_i with pairwise different slopes corresponds to a factorization $L = L_1 \cdots L_\nu$, $\Delta_{L_i} = \Delta_i$ in any order. The operators L_i are unique up to unit factors from $\mathbb{C}[\![t]\!]$ from left/right.

Remark on factorization of single-slope operators

If L is a single-slope and its Newton diagram is reducible (there are internal lattice points on the lower edge), then possibility of further factorization depends on arithmetic properties of roots of an auxiliary polynomial associated with the Δ-leading terms of L. Once again resonances appear and determine the reducibility.

A few references (Newton should be mentioned as a commutative precursor):

1. VAINBERG, M.M., TRENOGIN, V.A.: Theory of Branching of Solutions of Non-linear Equations, 1974.
2. VAN DER PUT, M., SINGER, M.F.: Galois Theory of Linear Differential Equations, 2003.
3. MEZUMAN, LEANNE, YAKOVENKO, S.: Formal Factorization of Higher Order Irregular Linear Differential Operators. *Arnold Math J.* , 2018.

A few words about the proof

Commutative case. Factorization in the algebra $\mathscr{O}(t)[\xi]$ or $\mathbb{C}[\![t]\!][\xi]$ with $[t,\xi]=0$. Solved by Newton (sliding ruler method). Fiat Newton polygons!

Geometrically: detection of irreducible components of a planar curve (germ).

Modern proof: desingularization (blow-up) on the plane $\mathbb{C}^2$.

Formal (Poincaré-Dulac-type) proof. Write the identity $L = MN$ as an infinite *triangular* system of equations in $\mathbb{C}[\xi]$ graded by powers of t and solve them inductively. Solvability of each equation is not at all straightforward, but it guaranteed by the "geometric" proof.

Noncommutative case: solvability of the infinite system follows from that of its commutative analog.

Reason: the "homological operators" survive the *mild non-commutativity* of $\mathbb{C}[\![t]\!][\epsilon]$: $[\epsilon, t] = t \ll 1$ (the formal parameter t is "small" cf. $[\partial, t] = 1$). Hence infinite system of equations over $\mathbb{C}[\epsilon]$ is inductively solvable.

Some action, finally?

Problem. F-classification of single-slope non-Fuchsian operators

Given a single-slope $L \in \mathscr{W} \smallsetminus \mathscr{F}$, find a simplest M such that $MH = GL$ for some $G, H \in \mathscr{F}$. Can one remove non-principal terms, like in $\mathscr{F}$?

1. The second condition $\text{r-gcd}(L, H) = 1$ becomes obsolete if L is non-Fuchsian (positive slope).
2. Once G is chosen, the product GL is "two-slope" and can be uniquely factored with the Fuchsian (zero slope) term being to the left or to the right. One factorization, GL, is the starting point, the other MH with Fuchsian H, yields M and H uniquely up to a unit in $\mathbb{C}[\![t]\!]$.
3. Abusing notation, we can "call" $H = \widehat{G}$ (of course, this "similarity" depends on L).
4. Ultimately, given L, we have an "action" $\widehat{\mathrm{Ad}}_{\bullet}$ of $\mathscr{F}$ on $\mathscr{W}$, defined as

$$L \xmapsto{\widehat{\mathrm{Ad}}_G} M = GL\widehat{G}^{-1} \quad (\text{to be understood as} \quad M\widehat{G} = GL).$$

An algebraist's nightmare

Pseudo-quasi-semi-action of $\mathscr{F}$ on $\mathscr{W}$

$$L \xmapsto{\widehat{\mathrm{Ad}}_G} M = GL\widehat{G}^{-1} \quad \text{(to be understood as} \quad M\widehat{G}^{-1} = GL).$$

- $\mathscr{F}$ not a group, only a semigroup. Notation G^{-1} is suggestive but misleading (although suggested by Ore himself). Besides, $G \neq \widehat{G}$.
- $\mathbb{C}[\![t]\!]$ is a graded (commutative) ring, but $\mathscr{W}$ is not a *graded* algebra, only *filtered* by powers of t (non-commutativity!).
- $\mathscr{F}$ is not a subalgebra in $\mathscr{W}$, only a multiplicatively closed subset.
- There is absolutely no idea how one could get from classification of single-slope operators to that of their compositions.

Intermediate conclusion. Apparently we lack some basic understanding of the Weyl-type algebras. Perhaps, an adequate algebraic structure hides beyond the above ad hoc constructions, but which one?

Teaser?

Perhaps, we need to extend the framework and consider instead of the polynomial $\mathbb{C}[\epsilon]$ the building brick

$$\mathscr{R} = \mathbb{C}[\epsilon][\![\epsilon^{-1}]\!] = \\ \cdots + c_{-n}\epsilon^{-k} + \cdots + c_{-1}\epsilon^{-1} + \\ c_0 + c_1\epsilon + \cdots + c_n\epsilon^n, \\ n < +\infty,$$

(Formal Laurent series in ϵ^{-1})?

Thinking in (slow) progress (with BORIS KHESIN). Too early even to informally discuss.

Cast



SHIRA, LEANNE and hopefully another future WIS M.Sc. student...

Thank you for your attention!