UNIVERSITY OF TORONTO
The Faculty of Arts and Science

DECEMBER 2016 EXAMINATIONS

MAT 244H1
Duration: 3 hours

NO AIDS ALLOWED

Total marks for this paper: 80.
This paper contains 2 pages.

1. (10 pts) Consider a Bernoulli equation

$$y' + \frac{2y}{x^2 - 1} = (x - 1)y^2.$$

   (a) (8 pts) Find the general solution of this equation in the interval $-1 < x < 1$.

   (b) (2 pts) How many solutions of this equation satisfy $y(-1) = 0$? How does this agree with the existence and uniqueness theorem?

2. (10 pts) Consider the equation

$$(3x^2 + y)dx + x(xy - 1)dy = 0.$$

   (a) (2 pts) Is this equation exact?

   (b) (4 pts) Find an integrating factor of the form $\mu(x)$.

   (c) (4 pts) Find the general solution. You may leave the solution in the implicit form.

3. (10 pts) Consider the equation

$$y'' + 2y' + 2y = \cos(x).$$

   (a) (8 pts) Find the general solution.

   (b) (2 pts) Find values of a, b such that the the solution of this equation satisfying $y(0) = a$, $y'(0) = b$ is bounded.

   (Recall that a function $f(x)$ is called bounded if there exists a constant C such that $|f(x)| < C$ for every x.)

4. (10 pts) Given that $y(x) = \dfrac{\sin(x)}{x}$ is a solution of

$$xy'' + 2y' + xy = 0,$$

solve the equation

$$xy'' + 2y' + xy = 1.$$

5. (10 pts) Consider the equation

$$y^{(4)} - y^{(2)} = (x+1)(e^{-x}+1) + 3(\sin(x)+1)e^{x}.$$

Use the undetermined coefficients method to write down the correct form of a particular solution. (You do not need to calculate the coefficients!)

6. (10 pts) Consider a linear system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & \varepsilon \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

   (a) (4 pts) For each $\varepsilon \neq 1$, determine the type and stability of the equilibrium point $(0, 0)$.

   (b) (4 pts) Sketch the phase portrait of the system for $\varepsilon = -1$ and $\varepsilon = 4$.

   (c) (2 pts) Sketch the phase portrait of the system for $\varepsilon = 1$. (Hint: Solve the system.)

7. (10 pts) Solve the initial value problem

$$\begin{cases} x' = x + 4y - 3 \\ y' = x + y \\ x(0) = -1 \\ y(0) = 3 \end{cases}.$$

8. (10 pts) Consider the system

$$\begin{cases} x' = x^2 - y - 2 \\ y' = (x - y)(x - y - 2). \end{cases}.$$

   (a) (2 pts) Find all its equilibrium points.

   (b) (4 pts) Determine the types and stability of these points.

   (c) (2 pts) Draw the phase portrait of the system near each of the equilibrium points.

   (d) (2 pts) Draw the global phase portrait.