Problem 1

This is a linear 1st order ODE, so we seek an integrating factor. For a linear equation $y' + q(x)y = g(x)$, the integrating factor is $\mu(x) = \exp\left(\int q(t)dt\right)$, so we compute:

$$\mu(x) = \exp\left(\int 4dt\right) = e^{4x}.$$

Multiplying both sides of the equation by μ, we get:

$$\begin{aligned} & e^{4x}y' + 4e^{4x}y = xe^{5x} \\ \implies \quad & \frac{d}{dx}\left(e^{4x}y\right) = xe^{5x} \\ \implies \quad & e^{4x}y = \int xe^{5x}dx + c \\ \implies \quad & e^{4x}y = \frac{e^{5x}(5x-1)}{25} + c \\ \implies \quad & y = \frac{e^{x}(5x-1)}{25} + ce^{-4x}. \end{aligned}$$

This is the general solution. To find the particular solution satisfying $y(0) = 1$, we plug in and solve for c:

$$\begin{aligned} 1 &= y(0) \\ &= \frac{e^0(-1)}{25} + ce^0 \\ &= \frac{-1}{25} + c\,, \end{aligned}$$

so $c = 1 + \frac{1}{25} = \frac{26}{25}$. Thus the final solution is:

$$y = \frac{e^{x}(5x-1)}{25} + \frac{26}{25}e^{-4x}.$$