The equation is presented as if it were exact, or could be made to be exact. We proceed under this assumption, and set $N(x, y) = xy + 1$ and $M(x, y) = y(x + y)$. We could check to see if the equation is already exact by testing whether M_y is equal to N_x (and in fact it is not), but the question gives us a hint that that is unnecessary by stating that an integrating factor of the form $\mu = \mu(y)$ exists.

We can calculate what this integrating factor would have to be by mimicking the book in Chapter 2.6, where the authors go over integrating factors that depend on x. If an integrating factor $\mu(y)$ existed, then after multiplying our equation by $\mu(y)$ the equation would be exact. That means that our new equation would be

$$\mu(y)M(x, y) + \mu(y)N(x, y)y' = 0,$$

and

$$(\mu(y)M(x, y))_y = (\mu(y)N(x, y))_x$$

by exactness. Expanding that last equation, we get

$$\frac{d\mu(y)}{dy}M + \mu(y)\frac{\partial M}{\partial y} = \mu(y)\frac{\partial N}{\partial x}.$$

One final rearrangement yields

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu(y).$$

If we let $Q = (N_x - M_y)/M$ then

$$Q = \frac{N_x - M_y}{M} = \frac{y - (x + 2y)}{y(x + y)} = \frac{-1}{y}.$$

This means that Q is a function of y alone, and

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu(y) = Q(y)\mu(y) = (-1/y)\mu(y)$$

from our earlier equation. This is a first order linear differential equation, so we can use integrating factors to get that

$$\mu(y) = e^{\int Q(y)\,\mathrm{d}y} = A/y$$

for some constant A. We're only trying to find one integrating factor, so we can set $A = 1$ for convenience and have $\mu(y) = 1/y$.

Another valid way to find the integrating factor is simply to use the formula presented in the textbook for the integrating factor. The result is the same. It should be pointed out that a lot of people tried to use an incorrect formula for $\mu(y)$ and wrote that

$$\mu(y) = \frac{N_x - M_y}{M} = \frac{-1}{y}.$$

This formula will not work in general, but it did (coincidentally) give a valid integrating factor in this problem.

Now that we have the integrating factor we can multiply through the differential equation by $\mu(y)$ and get

$$x + y + (x + 1/y)y' = 0.$$

If we wanted we could test this equation to make sure it is exact as a check on our work. Since it is exact, there must be a function $\psi(x, y)$ satisfying

$$\psi_x = x + y$$

and

$$\psi_y = x + 1/y$$

whose level curves are the solutions to the differential equation. Integrating both sides of the first equation by x we get

$$\psi(x, y) = \int x + y \, \mathrm{d}x + h(y) = \frac{x^2}{2} + xy + h(y)$$

for some unknown function h depending only on y. Using the second equation we see that

$$x + h'(y) = \psi_y = x + 1/y,$$

so

$$h'(y) = 1/y,$$

and consequently $h(y) = \log(|y|) + C$ for some constant C. Since we only care about ψ up to a constant we can set $C = 0$. So

$$\psi(x, y) = \frac{x^2}{2} + xy + \log(|y|)$$

is a function whose level curves $\psi(x, y(x)) = C$ are the solutions of the differential equation (given implicitly). This is the general solution, and since there is no obvious way to solve for y in terms of x we leave it in implicit form.

Since we are given an initial condition $y(0) = e$ we know that for that particular solution,

$$C = \psi(0, y(0)) = \psi(0, e) = \frac{0^2}{2} + (0)e + \log(|e|) = 1.$$

So the solution satisfying the initial condition $y(0) = e$ is given implicitly by

$$1 = \frac{x^2}{2} + xy + \log(|y|).$$

If we notice that y can never be zero (otherwise the logarithm would be undefined) and $y(0) = e$ is positive, we can say that

$$1 = \frac{x^2}{2} + xy + \log(y)$$

on the solution interval that contains 0.