Problem 3

Consider the following BERNOUILLI's problem

$$\begin{cases} y' + y = r(x)y^2 \\ y(0) = 1 \end{cases} \quad \text{where} \quad r(x) = \begin{cases} 1, & \text{if } x \le 1 \\ 0, & \text{if } x > 1 \end{cases} \tag{1}$$

1. Find a solution for $x \le 1$.
2. Assuming that the solution is continuous at $x = 1$, use the result of part 1 to find the solution to the problem for $x > 1$.

Solution

1. On $x \le 1$ the IVP is

$$\begin{cases} y' + y = y^2 \\ y(0) = 1 \end{cases} \tag{2}$$

Use the *Bernouilli substitution* $u = y^{-1}$ to reduce the nonlinear differential equation to a linear one: divide through by $-y^2$ and substitute $u' = -y^{-2}y'$ to get

$$\begin{aligned} & -y'y^{-2} - y^{-1} = -1 \\ \iff \quad & u' - u = -1 \end{aligned}$$

Thus the IVP becomes

$$\begin{cases} u' - u = -1 \\ u(0) = 1 \end{cases} \tag{3}$$

This is an equation we know how to solve. You may use the "integrating factor" method, or the "variation of parameter method." I will solve it using the variation of parameter method.

- solutions of the associated homogeneous differential equation $u' - u = 0$ are of the form $u(x) = ae^x$ for $a =$ a real number
- vary the parameter $a = a(x)$, and work backwards: assume ae^x satisfies the given inhomogeneous differential equation

$$\begin{aligned} -1 &= (ae^x)' - (ae^x) \\ &= a'e^x + \cancel{ae^x} - \cancel{ae^x} \\ &= a'e^x \end{aligned}$$

  getting a differential equation for a
- rearrange it and integrate to solve for a:

$$\begin{aligned} & a' = -e^{-x} \\ \Rightarrow \quad & \int a' \mathsf{d}x = \int -e^{-x} \mathsf{d}x \\ \Rightarrow \quad & a = e^{-x} + c \end{aligned}$$

where $c =$ a constant (byproduct of integration)

- thus

$$u(x) = ae^x = (e^{-x} + c)e^x = 1 + ce^x$$

  is a solution of (3) iff c satisfies the initial condition

- solve for the c that does: $1 = u(0) = 1 + ce^0 = 1 + c$ forcing $c = 0$ and $u(x) = 1$

Conclude $y(x) = u(x)^{-1} = 1$ also.

2. The solution to the two-piece discontinuous IVP (1) is also made up of two pieces! Each piece is the solution to an IVP. We just found the $x \leq 1$ piece by solving the IVP (2). What IVP should we solve to find the $x > 1$ piece? Well we know what the differential equation is on $x > 1$. But, what should the initial condition be? The assumption that the two-piece solution should be continuous at $x = 1$ forces us to put the initial condition $y(1) = 1$. That is, the $x > 1$ piece of the continuous solution we seek is the solution to the the IVP

$$\begin{cases} y' + y = 0 \\ y(1) = 1 \end{cases} \tag{4}$$

The equation $y' + y = 0$ requires no special methods. Just rearrange and integrate to find $y = ce^{-x}$. Solve for c by plugging in the initial condition $1 = y(1) = ce^{-1}$ forcing $c = e$.

$$\boxed{y(x) = \begin{cases} 1, & \text{if } x \leq 1 \\ e^{-x+1}, & \text{if } x > 1 \end{cases}} \tag{5}$$