Problem 5

Put $X = x + 1$ and $Y = y + 1$. Then

$$\frac{dy}{dx} = \frac{dY}{dx} \div \frac{dY}{dy} = \left(\frac{dY}{dX} \cdot \frac{dX}{dx}\right) \div \frac{dY}{dy} = \frac{dY}{dX}.$$

Substituting this into the given ODE, we get

$$\frac{dY}{dX} = \frac{Y - X}{Y + X} \Longleftrightarrow \frac{dY}{dX} = \frac{Y/X - 1}{Y/X + 1}$$

This is a homogeneous equation. Put $u = Y/X$, then

$$Y = uX \Rightarrow \frac{dY}{dX} = \frac{du}{dX}X + u.$$

So,

$$\frac{du}{dX}X + u = \frac{u-1}{u+1} \Longleftrightarrow \frac{du}{dX}X = \frac{u-1}{u+1} - u \Longleftrightarrow \frac{du}{dX}X = \frac{-u^2-1}{u+1}$$

$$\Longleftrightarrow -\frac{(u+1)du}{u^2+1} = \frac{dX}{X} \Longleftrightarrow \int -\frac{(u+1)}{u^2+1}\,du = \int \frac{dX}{X} \Longleftrightarrow$$

$$-\arctan u - \frac{1}{2}\ln(u^2+1) = \ln|X| + C \Longleftrightarrow -\arctan\frac{y+1}{x+1} - \frac{1}{2}\ln\left[\left(\frac{y+1}{x+1}\right)^2 + 1\right] = \ln|x+1| + C$$