MAT244H1F1: INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

SECOND MIDTERM SOLUTIONS

Problem 1. *Solve the initial value problem*

$$\begin{aligned} y'' - y' - 6y &= 6 \\ y(0) &= 1, \\ y'(0) &= 1. \end{aligned}$$

Solution: The homogeneous linear equation is $y'' - y' - 6y = 0$ which has characteristic equation $r^2 - r - 6 = 0$. We can factor this as $(r-3)(r+2) = 0$, and so we find that two generating solutions are $y(t) = e^{3t}$ and $y(t) = e^{-2t}$. Hence all solutions to this equation are of the form

$$y(t) = Ae^{3t} + Be^{-2t},$$

for some constants A and B. Moreover, it is clear that a particular solution to $y'' - y' - 6y = 6$ is the constant solution $y(t) = -1$, from where we deduce all solutions to this equation are of the form

$$y(t) = Ae^{3t} + Be^{-2t} - 1.$$

Notice that the constant solution $y(t) = -1$ can be obtained by the method of undetermined coefficients since 0 is not a root of the characteristic polynomial and the right hand side is a constant.

Differentiating the general solution we find

$$y'(t) = 3Ae^{3t} - 2Be^{-2t}.$$

Using the initial conditions we obtain a linear system in A and B:

$$\begin{aligned} A + B - 1 &= 1, \\ 3A - 2B &= 1, \end{aligned}$$

which has a unique solution $A = 1$ and $B = 1$. In conclusion, the solution we look for is

$$y(t) = e^{3t} + e^{-2t} - 1.$$

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Problem 2. **(a):** *Find a solution of the equation*

$$x^2y'' - x(x+2)y' + (x+2)y = 0.$$

of the form $y = ax + b$, where a and b are constants.

(b): *Find the general solution of the equation*

$$x^2y'' - x(x+2)y' + (x+2)y = 2x^3e^{2x}.$$

Solution: By substituting directly we find

$$0 = -x(x+2)a + (x+2)(ax+b) = (x+2)(ax+b-ax) = b(x+2),$$

from where we conclude $b = 0$ and there is no restriction in a. For example, $y_1(x) = x$ is one such solution.

To find another solution we use D'Alambert's method: write $y(x) = u(x)x$ and notice that

$$y' = u'x + u$$
$$y'' = u''x + 2u'.$$

We substitute into the equation to find the equation u needs to satisfy:

$$\begin{aligned} 0 &= x^2(u''x + 2u') - x(x+2)(u'x+u) + (x+2)ux \\ &= u''x^3 + 2u'x^2 - u'x^3 - 2u'x^2 \\ &= (u'' - u')x^3, \end{aligned}$$

from where we deduce u must satisfy $u'' = u'$, which is satisfied, for example, by $u(x) = e^x$. Hence $y_2(x) = xe^x$ is another solution. The Wronskian of y_1, y_2 is:

$$x(x+1)e^x - xe^x = x^2e^x \neq 0,$$

and so they are a fundamental pair of solutions. Further, write the non-homogeneous equation as

$$y'' - \frac{(x+2)}{x}y' + \frac{(x+2)}{x^2}y = 2xe^{2x}$$

and apply the variation of parameters formula to obtain the general solution:

$$\begin{aligned} Y(x) &= -x\int \frac{(xe^x)(2xe^{2x})}{x^2e^x}\mathrm{d}x + xe^x\int \frac{(x)(2xe^{2x})}{x^2e^x}\mathrm{d}x = \\ &= -x\int 2e^{2x}\mathrm{d}x + xe^x\int 2e^x\mathrm{d}x = \\ &= -xe^{2x} + Ax + 2xe^{2x} + Bxe^x = xe^{2x} + Ax + Bxe^x. \end{aligned}$$

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Problem 3. **(a):** *Find the general solution of*

$$y^{(6)} - y^{(2)} = 0.$$

(b): *Find the general solution of*

$$y^{(6)} - y^{(2)} = x.$$

Solution: The characteristic equation of the homogeneous problem is $r^6 - r^2 = 0$. This factors as

$$r^2(r-1)(r+1)(r+i)(r-i) = 0,$$

from where we deduce the solutions 1 and x (corresponding to the repeated root 0), e^x, e^{-x} (corresponding to the roots 1 and -1), and $\cos(x)$, $\sin(x)$ (corresponding to the complex conjugate roots i and $-i$). Hence the general solution is of the form

$$y(x) = A + Bx + Ce^x + De^{-x} + E\cos(x) + F\sin(x).$$

To find a particular solution we notice that 0 is a double root and so there is double resonance from where we deduce there must be a solution of the form

$$y(x) = (\alpha + \beta x)x^2 = \alpha x^2 + \beta x^3.$$

Notice that the sixth derivative of $y(x)$, vanishes, and the second derivative is $y''(x) = 2\alpha + 6\beta x$. Hence the equation becomes

$$-2\alpha - 6\beta x = x,$$

from where we deduce, comparing coefficients, that $\alpha = 0$ and $\beta = \frac{1}{6}$. We conclude a particular solution is $y(x) = -\frac{x^3}{6}$ and so the general solution to the non homogeneous equation is

$$y(x) = A + Bx + Ce^x + De^{-x} + E\cos(x) + F\sin(x) - \frac{x^3}{6},$$

where A, B, C, D, E, F are constants.

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Problem 4. **(a):** *Solve the equation*

$$y''' + y' = \cos(kx),$$

where k is a constant.

(b): *For which values of k does the equation in part (a) has an unbounded solution?*

Solution: The characteristic equation of the homogeneous equation is $r^3 + r = 0$ which is factored as $r(r+i)(r-i) = 0$. Hence we conclude 1, $\sin(x)$, and $\cos(x)$ are solutions of the homogeneous equation and hence the general solution is

$$y(x) = A + B\cos(x) + C\sin(x).$$

We will use the method of undetermined coefficients to find a particular solution of this equation. First suppose that $k \neq 0, 1, -1$. Since in the right hand side we have $\cos(kx)$, and k is not a root of the characteristic equation, the form of a particular solution is

$$y(x) = a\cos(kx) + b\sin(kx).$$

We have

$$\begin{aligned} y'(x) &= -ak\sin(kx) + bk\cos(kx), \\ y''(x) &= -ak^2\cos(kx) - bk^2\sin(kx), \\ y'''(x) &= ak^3\sin(kx) - bk^3\cos(kx). \end{aligned}$$

Substituting we have

$$\begin{aligned} \cos(kx) = y''' + y' &= ak^3\sin(kx) - bk^3\cos(kx) - ak\sin(kx) + bk\cos(kx) \\ &= a(k^3 - k)\sin(kx) - b(k^3 - k)\cos(kx). \end{aligned}$$

Hence, since $k^3 - k \neq 0$, we can take $a = 0$ and $b = -\frac{1}{k^3 - k}$ and obtain as particular solution $y(x) = -\frac{\sin(kx)}{k^3 - k}$. We conclude the general solution is

$$y(x) = A + B\cos(x) + C\sin(x) - \frac{\sin(kx)}{k^3 - k}.$$

Notice that in this case

$$|y(x)| = |A + B\cos(x) + C\sin(x) - \frac{\sin(kx)}{k^3 - k}| \leq |A| + |B| + |C| + |(k^3 - k)|^{-1},$$

which is bounded.

Three cases remain: $k = 0, 1, -1$. For the case $k = 0$ the right hand side is $\cos(0) = 1$ and, since 0 is a simple root of the characteristic equation, we conclude there must be a solution of the form $y(x) = ax$. Substituting we get

$$1 = y''' + y' = 0 + a,$$

hence $y(x) = x$ is a particular solution and the general solution in this case is

$$y(x) = A + B\cos(x) + C\sin(x) + x,$$

and this is unbounded.

For $k = \pm 1$ notice that the right hand side is $\cos(\pm x) = \cos(x)$ since $\cos(x)$ is an even function. In this case the particular solution has the form

$$y(x) = ax\cos(x) + bx\sin(x).$$

Notice that

$$\begin{aligned} y'(x) &= a\cos(x) - ax\sin(x) + b\sin(x) + bx\cos(x), \\ y''(x) &= -a\sin(x) - a\sin(x) - ax\cos(x) + b\cos(x) + b\cos(x) - bx\sin(x) \\ &= -2a\sin(x) + 2b\cos(x) - ax\cos(x) - bx\sin(x) \\ y'''(x) &= -2a\cos(x) - 2b\sin(x) - a\cos(x) + ax\sin(x) - b\sin(x) - bx\cos(x) \\ &= -3a\cos(x) - 3b\sin(x) + ax\sin(x) - bx\cos(x). \end{aligned}$$

Hence, when we substitute,

$$\begin{aligned} \cos(x) &= y'''(x) + y'(x) \\ &= -3a\cos(x) - 3b\sin(x) + ax\sin(x) \\ &\quad - bx\cos(x) + a\cos(x) - ax\sin(x) + b\sin(x) + bx\cos(x) \\ &= -2a\cos(x) - 2b\sin(x), \end{aligned}$$

from where we conclude $a = -\dfrac{1}{2}$ and $b = 0$. In other words a particular solution is $y(x) = -\dfrac{x\cos(x)}{2}$, and so the general solution is

$$y(x) = A + B\cos(x) + C\sin(x) - \frac{x\cos(x)}{2},$$

which again is unbounded.

In conclusion, the general solutions are

$$y(x) = A + B\cos(x) + C\sin(x) - \frac{\sin(kx)}{k^3 - k}, k \neq 0, 1, -1,$$

and

$$y(x) = A + B\cos(x) + C\sin(x) + x, k = 0,$$

and

$$y(x) = A + B\cos(x) + C\sin(x) - \frac{x\cos(x)}{2}, k = \pm 1,$$

and there are unbounded solutions only when $k = 0, 1, -1$.

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