MAT337H1, Introduction to Real Analysis: additional recommended problems for Feb 10 class

1. Fill in the gaps in the proof of the continuity criterion in terms of sequences (first theorem proved in Feb 10 class). Namely, explain why in the proof of the implication (b) $\Rightarrow$ (a) one has that

   - the sequence x_n converges to x_0;
   - the sequence $f(x_n)$ does not converge to $f(x_0)$.

2. Let g be a function defined on $S \subset \mathbb{R}$, and let f be a function defined on $g(S) = \{g(x) \mid x \in S\}$. Assume that g is continuos at $x_0 \in S$, and f is continuos at $g(x_0)$. Prove that $f(g(x))$ is continuos at x_0. (In other words, the composition of two continuos functions is continuos.)

3. Let
$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$
(This is known as the Dirichlet function.) Prove that this function is nowhere continuos (i.e., it is not continuos at any point).

4. Let f be a function defined in some interval (a, b) containing a point x_0. Assume also that f is continuos at x_0, and $f(x_0) \neq 0$. Prove that there exists $\delta > 0$ such that $f(x)$ has the same sign as $f(x_0)$ whenever $|x - x_0| < \delta$.

5. Let $f, g \colon \mathbb{R} \to \mathbb{R}$ be continuos functions (this means that they are continuos for every $x \in \mathbb{R}$). Prove that the function $\max(f, g)$, defined by
$$\max(f, g)(x) = \max(f(x), g(x)),$$
is continuos.