## MAT337H1, Introduction to Real Analysis: solution to Exercise H from Section 5.3

We need to show that $h(x)$ is continuous at every point $x_0 \in [a, c]$. Notice that if $x_0 < b$, then there is neighborhood of $x_0$ where $h = f$, and since $f$ is continuos at $x_0$, so is $h$. (Check that if two functions coincide in a neighborhood of $x_0$, and one of them is continuos at $x_0$, then the other one is also continuos. This can be reformulated by saying that continuity at a point is a local property, determined only by values of the function near that point.) Analogously, if $x_0 > b$, then there is neighborhood of $x_0$ where $h = g$, and since $g$ is continuos at $x_0$, so is $h$. So, it suffices to show that $h$ is continuous at $b$. Take any $\varepsilon > 0$. Then, since $f$ is continuous at $x_0$, there exists $\delta > 0$ such that $|f(x) - f(b)| < \varepsilon$ whenever $|x - b| < \delta$. Furthermore, since $g$ is continuous at $b$, there exists $\delta' > 0$ such that $|g(x) - g(b)| < \varepsilon$ whenever $|x - b| < \delta'$. Take $\delta'' = \min(\delta, \delta')$. Then, for $|x - b| < \delta''$ we have both $|f(x) - f(b)| < \varepsilon$ and $|g(x) - g(b)| < \varepsilon$. If we further assume that $x \leq b$, we get

$$|h(x) - h(b)| = |f(x) - f(b)| < \varepsilon,$$

while for $x > b$ we have

$$|h(x) - h(x_0)| = |g(x) - g(b)| < \varepsilon.$$

So, $|h(x) - h(b)| < \varepsilon$ provided that $|x - b| < \delta''$, and continuity of $h$ at $b$ follows.