MAT337H1, Introduction to Real Analysis: solution to Exercise C for Section 5.6 and Problem 1 from additional recommended problems for Feb 15 class

Exercise C for Section 5.6. Show that $2\sin(x) + 3\cos(x) = x$ has three solutions.

Solution. Consider the function $f(x) = 2\sin(x) + 3\cos(x) - x$. Notice that

$$f(-\pi) = -3 + \pi > 0, \quad f(-\frac{\pi}{2}) = -2 + \frac{\pi}{2} < 0, \quad f(0) = 3 > 0, \quad f(\pi) = -3 - \pi < 0.$$

So, the function f changes sign in intervals $[-\pi, -\frac{\pi}{2}]$, $[-\frac{\pi}{2}, 0]$, $[0, \pi]$, and hence has at least one zero in each of these intervals. (Here we are using continuity of f and the intermediate value theorem.) But $f(x) = 0$ if and only if $2\sin(x) + 3\cos(x) = x$, so it follows that the latter equation has at least three solutions.

Showing that the our equation has exactly three solutions is more tricky. First, using some trigonometry, we get

$$2\sin(x) + 3\cos(x) = \sqrt{15}(\sin(x)\cos(\alpha) + \cos(x)\sin(\alpha)) = \sqrt{15}\sin(x + \alpha),$$

where α is such that $\cos(\alpha) = 2/\sqrt{15}$ and $\sin(\alpha) = 3/\sqrt{15}$. It follows that the left-hand side of our equation is always between $\sqrt{15}$ and $-\sqrt{15}$, which means that all solutions of the equation belong to the interval $[-\sqrt{15}, \sqrt{15}]$. Further, notice that for $x \in [\pi, \frac{3\pi}{2}]$ the left-hand side of our equation is non-positive, while the right-hand side is positive, so there are no solutions in this interval. Since $\sqrt{15} \in [\pi, \frac{3\pi}{2}]$, it follows that all solutions of our equation in fact lie in the interval $[-\sqrt{15}, \pi]$. Further, if $x \in [-\frac{3\pi}{2}, -\pi]$, we have $\sin(x) \geq 0$, so $2\sin(x) + 3\cos(x) \geq -3$. On the other hand, the right-hand side of our equation in this interval is less than -3. So, there are no solutions in $[-\frac{3\pi}{2}, -\pi]$ as well, and all solutions lie in $[-\pi, \pi]$.

Now it remains to show that $f(x)$ has at most three zeros in the interval $[-\pi, \pi]$. Using the above computation, we have

$$f(x) = \sqrt{15}\sin(x + \alpha) - x,$$

so

$$f'(x) = \sqrt{15}\cos(x + \alpha) - 1.$$

Notice that the solutions of the equation $f'(x) = 0$ are of the form $x_1 + 2\pi k$, $x_2 + 2\pi k$. Therefore, $f'(x)$ has at most two zeros in $(-\pi, \pi)$. (One zero of the form $x_1 + 2\pi k$, and one of the form $x_2 + 2\pi k$. There cannot be two zeros of the same form, because then the distance between them would be at least 2π, so they cannot both lie in $(-\pi, \pi)$.) But then it follows that f has at most three zeros in $[-\pi, \pi]$: if there were four zeros, it would follow by Rolle's theorem that $f'(x)$ has at least three zeros in $(-\pi, \pi)$. (An alternative approach not employing Rolle's theorem is to use that f is monotonous in the intervals where f' does not change sign, and to use that a strictly monotonous function cannot take the same value twice.)

Problem 1 from additional recommended problems. In class we proved that if f is a continuous function on $[a, b]$, and ξ is a number such that $f(a) < \xi < f(b)$, then there is $c \in [a, b]$ such that $f(c) = \xi$. We defined c by the formula $c = \sup\{x \in [a, b] \mid f(x) < \xi\}$.

Then we showed that $f(c) = \xi$ by considering two cases $f(c) > \xi$ and $f(c) < \xi$ and drawing a contradiction in both cases. However, our argument does not work if $c = a$ in the first case or $c = b$ in the second case. Show that both these situations are in fact impossible.

Solution. Continuity of f at b means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(b)| < \varepsilon$ whenever $|x - b| < \delta$ and $x \in [a, b]$. Take $\varepsilon = f(b) - \xi$ (this is a positive number), and find the corresponding δ. Then for $x \in (b-\delta, b]$ we have $|f(x)-f(b)| < f(b)-\xi$, which in particular means that $f(x) > \xi$. So, all points of $(b - \delta, b]$ are not in the set $S = \{x \in [a, b] \mid f(x) < \xi\}$, and it follows that $b - \delta$ is an upper bound for S. So, for the least upper bound c, we have $c \leq b - \delta$, and thus $c \neq b$.

Further, continuity of f at a means that for any $\varepsilon > 0$ there exists $\delta' > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$ and $x \in [a, b]$. Take $\varepsilon = \xi - f(a)$ (this is a positive number), and find the corresponding δ'. Then for $x \in [a, a + \delta')$ we have $|f(x) - f(a)| < \xi-f(a)$, which in particular means that $f(x) < \xi$. So, $[a, a+\delta') \subset S = \{x \in [a, b] \mid f(x) < \xi\}$. In particular, we have $a + \frac{1}{2}\delta' \in S$. But since c is an upper bound for S, it follows that $c \geq a + \frac{1}{2}\delta'$, and thus $c \neq a$, ending the proof.