MAT337H1, Introduction to Real Analysis: solution of Problems 1b and 1c from additional recommended problems for Feb 3 class

Problem 1b. Show, using the definition of continuity, that the function $f(x) = \frac{1}{x}$ is continuous at every point where it is defined.

Solution. This function is defined for any $x \neq 0$. So we need to show that it is continuous at every point $x_0 \neq 0$. Continuity at x_0 means that for any $\varepsilon > 0$ there is $\delta > 0$ such that $|\frac{1}{x} - \frac{1}{x_0}| < \varepsilon$ whenever $|x - x_0| < \delta$ (and $x \neq 0$). We have

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x - x_0|}{|x||x_0|}.$$

Provided that $|x - x_0| < \delta$, the numerator is bounded above by δ. Further, we need to bound the denominator from below. Since $|x_0|$ is a constant, it suffices to bound below $|x|$. Although there is no universal bound ($|x|$ can take any non-zero value), $|x|$ is bounded below when x is sufficiently close to x_0. For example, if $\delta \leq \frac{1}{2}|x_0|$, we have that $|x| > \frac{1}{2}|x_0|$ whenever $|x - x_0| < \delta$. So, if $\delta \leq \frac{1}{2}|x_0|$ and $|x - x_0| < \delta$, we get

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| < \frac{2\delta}{x_0^2},$$

and to get that $|\frac{1}{x} - \frac{1}{x_0}| < \varepsilon$, we need

$$\frac{2\delta}{x_0^2} \leq \varepsilon.$$

Combining this with the inequality $\delta \leq \frac{1}{2}|x_0|$, we conclude that

$$\delta = \min\left(\frac{|x_0|}{2}, \frac{\varepsilon x_0^2}{2}\right)$$

is what we are looking for.

Problem 1c. Show, using the definition of continuity, that the function $f(x) = \sqrt{x}$ is continuous at every point where it is defined.

Solution. This function is defined for any $x \geq 0$. So we need to show that it is continuous at every point $x_0 \geq 0$. We do it separately for $x_0 \neq 0$ and $x_0 = 0$. First consider the case $x_0 \neq 0$. Continuity at x_0 means that for any $\varepsilon > 0$ there is $\delta > 0$ such that $|\sqrt{x} - \sqrt{x_0}| < \varepsilon$ whenever $|x - x_0| < \delta$ (and $x \geq 0$). For $x_0 \neq 0$, we have

$$|\sqrt{x} - \sqrt{x_0}| = \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} \leq \frac{|x - x_0|}{\sqrt{x_0}},$$

so if $|x - x_0| < \varepsilon\sqrt{x_0}$, then $|\sqrt{x} - \sqrt{x_0}| < \varepsilon$. Hence we can take $\delta = \varepsilon\sqrt{x_0}$.

Now we prove continuity at $x_0 = 0$. Continuity at 0 means that for any $\varepsilon > 0$ there is $\delta > 0$ such that $|\sqrt{x}| < \varepsilon$ whenever $|x| < \delta$ (and $x \geq 0$). Clearly, $\delta = \varepsilon^2$ works. (Indeed, $0 \leq x < \varepsilon^2 \Rightarrow \sqrt{x} < \varepsilon$.) So, continuity at 0 is also proved.