UNIVERSITY OF TORONTO
The Faculty of Arts and Science

APRIL 2017 EXAMINATIONS

MAT 337H1
Duration: 3 hours

NO AIDS ALLOWED

Total marks for this paper: 100.
This paper contains 2 pages.

Note: It is not necessary to reprove facts which are proved in the textbook or have been proved in class. However, you should always state the fact that you are using. When using a theorem, say under what conditions the theorem applies and explain why these conditions are satisfied in your case.

1. (a) (10 pts) Prove that
$$\lim_{n\to\infty} \frac{n+2}{2n-3} = \frac{1}{2}.$$
   (b) (10 pts) Does there exist natural number N such that
$$\left| \frac{n+2}{2n-3} - \frac{1}{2} \right| < 10^{-5}$$
   for every natural number $n \geq N$? If yes, find this number N. If no, prove that it does not exist.

2. (10 pts) Prove that the equation $x^5 + x = a$ has a real solution for any $a \in \mathbb{R}$.
   Hint: Apply the intermediate value theorem on a suitable interval.

3. (a) (10 pts) Define what it means for a function f to be Riemann integrable on $[a, b]$ and for a number I to be the integral of f on $[a, b]$.
   (b) (10 pts) Let f be a function integrable on $[0, 2]$. Show, using the definition from part (a), that $f(2x)$ is integrable on $[0, 1]$ and that
$$\int_0^1 f(2x)dx = \frac{1}{2}\int_0^2 f(x)dx.$$
   Hint: If $P = \{x_0 < \cdots < x_n\}$ is a partition of $[0, 1]$, and $X = \{x_1^*, \ldots, x_n^*\}$ is an evaluation sequence for P, then $P' = \{2x_0 < \cdots < 2x_n\}$ is a partition of $[0, 2]$, and $X' = \{2x_1^*, \ldots, 2x_n^*\}$ is an evaluation sequence for P'. Compare the Riemann sums $I(f(2x), P, X)$ and $I(f(x), P', X')$.

4. (10 pts) Let $f(x)$ be a function which is continuous for any $x \in \mathbb{R}$ and differentiable for any $x \neq 0$. Assume also that there exists a (finite) limit $\lim_{x\to 0} f'(x)$. Prove that $f(x)$ is differentiable at 0, and that $f'(0) = \lim_{x\to 0} f'(x)$.

   Hint: Use the definition of $f'(0)$. Apply the mean value theorem to the numerator of the expression under the limit sign.

5. (a) (10 pts) Let (X_1, ρ_1) and (X_2, ρ_2) be metric spaces. Define what it means for a mapping $\phi\colon X_1 \to X_2$ to be continuous (with respect to the metrics ρ_1, ρ_2).

   (b) (10 pts) Let X_1 be the set $C[0, 1]$ of functions continuous on $[0, 1]$ endowed with the L^1-metric

$$\rho_1(f, g) = \int_0^1 |f(x) - g(x)| dx.$$

   Let also X_2 be the same set endowed with the uniform metric

$$\rho_2(f, g) = \sup_{[0,1]} |f - g|.$$

   Let ϕ be the identity mapping from X_1 to X_2, i.e., the mapping given by $\phi(f) = f$. Is ϕ continuous? Justify your answer.
   Hint: Find a sequence of functions $f_n \in C[0, 1]$ converging to 0 in the L^1-metric but not converging to 0 in the uniform metric.

6. Let a_n be a bounded sequence of real numbers, and let $b_n = \sup\{a_n, a_{n+1}, \dots\}$.

   (a) (10 pts) By using the monotone convergence theorem, or otherwise, show that the sequence b_n is convergent.

   (b) (10 pts) Assume, in addition, that a_n is convergent. Show that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.