MAT337H1, Introduction to Real Analysis: notes on Riemann integration

1 Definition of the Riemann integral

Definition 1.1. Let $[a, b] \subset \mathbb{R}$ be a closed interval. A *partition* P of $[a, b]$ is a finite set of points $x_0 < x_1 < \cdots < x_{n-1} < x_n$ in $[a, b]$ such that $x_0 = a$ and $x_n = b$.

Each such collection of points partitions $[a, b]$ into subintervals $[x_0, x_1], \ldots, [x_{n-1}, x_n]$. Hence the name.

Let $P = \{x_0 < x_1 < \cdots < x_{n-1} < x_n\}$ be a partition of $[a, b]$. Denote the interval $[x_{j-1}, x_j]$ by I_j. We get n such intervals $I_1, \ldots, I_n$.

Definition 1.2. A set $X = \{x'_1, \ldots, x'_n\}$ of n real numbers is called an *evaluation sequence* for a partition $P = \{x_0 < x_1 < \cdots < x_{n-1} < x_n\}$ if $x'_j \in I_j$ for every $j = 1, \ldots, n$.

Further, let f be a function on $[a, b]$. Let also $\Delta_j = x_j - x_{j-1}$ be the length of the interval I_j.

Definition 1.3. The *Riemann sum* associated with a function f on $[a, b]$, a partition P of $[a, b]$, and an evaluation sequence $X = \{x'_1, \ldots, x'_n\}$ for partition P is the number

$$I(f, P, X) = \sum_{j=1}^{n} f(x'_j)\Delta_j.$$

The number $I(f, P, X)$ can be interpreted as the (signed) area between the horizontal axis and the graph of a piecewise constant function equal to $f(x'_j)$ on the interval I_j. This function is a good approximation for f when (adjacent) points of P are close to each other. For this reason, we would like to define the integral of f (i.e., the area between the horizontal axis and the graph of f) as the limit of $I(f, P, X)$, as the points of P get close to each other. To define this limit, we introduce the following notion.

Definition 1.4. The *mesh* of a partition $P = \{x_0 < x_1 < \cdots < x_{n-1} < x_n\}$ is the number $\mathrm{mesh}(P)$ defined by $\mathrm{mesh}(P) = \max(\Delta_1, \ldots, \Delta_n)$.

In other words, the mesh is the maximal distance between adjacent points of the partition. The mesh of a partition P is small if and only if all adjacent points of P are close to each other. Thus, we define the integral of f to be the limit $\lim_{\mathrm{mesh}(P)\to 0} I(f, P, X)$. This does not yet have a precise meaning, because $I(f, P, X)$ is not a function of $\mathrm{mesh}(P)$. Instead, it depends on P itself, as well as on X. A precise definition of the integral is the following.

Definition 1.5. A function f on $[a, b]$ is called *(Riemann) integrable* on $[a, b]$ if there is a number $I \in \mathbb{R}$ with the following property: for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any partition P of $[a, b]$ with $\mathrm{mesh}(P) < \delta$ and any evaluation sequence X we have $|I(f, P, X) - I| < \varepsilon$. The number I is called the *(Riemann) integral*[1] of f on $[a, b]$ and is denoted by $\int_a^b f(x)dx$.

[1] Check that if the number I with the above property exists, then it is unique. Therefore, any integrable function has well-defined integral.

Example 1.6. The Dirichlet function

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

is not integrable on $[0, 1]$. Indeed, let P be any partition of $[0, 1]$. Then, since any interval contains both rational and irrational numbers, we can choose an evaluation sequence X_1 all whose points are rational, and an evaluation sequence X_2 all whose points are irrational. Then the corresponding Riemann sums are $I(f, P, X_1) = 1$ and $I(f, P, X_2) = 0$. On the other hand, if f is integrable, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any partition P of $[a, b]$ with $\text{mesh}(P) < \delta$ and any evaluation sequence X we have $|I(f, P, X) - I| < \varepsilon$, where $I = \int_0^1 f(x)dx$. Applying this for $\varepsilon = \frac{1}{2}$, any partition P of $[a, b]$ with $\text{mesh}(P) < \delta$, and X_1, X_2 constructed above, we get that $|1 - I| < \frac{1}{2}$ and $|I| < \frac{1}{2}$. But numbers I with these properties do not exist. So, the Dirichlet function is not integrable.

2 Integrability of continuous functions

In this section we prove the following result.

Theorem 2.1. *Every function continuous on a closed interval* $[a, b]$ *is integrable on* $[a, b]$.

To prove this theorem, we need several preliminary statements. First, we introduce the notions of *lower and upper sums*:

Definition 2.2. Let f be a bounded function on a closed interval $[a, b]$. For a partition $P = \{x_0 < x_1 < \cdots < x_{n-1} < x_n\}$ of $[a, b]$ the corresponding *upper sum* is

$$U(f, P) = \sum_{j=1}^{n} (\sup_{x \in I_j} f(x)) \cdot \Delta_j,$$

where, as above, $I_j = [x_{j-1}, x_j]$, and $\Delta_j = x_j - x_{j-1}$ is the length of the interval I_j. Similarly, the *lower sum* corresponding to f and P is

$$L(f, P) = \sum_{j=1}^{n} (\inf_{x \in I_j} f(x)) \cdot \Delta_j.$$

Comparing this definition with the definition of Riemann sums, we get the following.

Proposition 2.3. *Let* f *be a bounded function on a closed interval* $[a, b]$. *Then, for any partition* P *of* $[a, b]$ *and any evaluation sequence* X *for* P, *we have*

$$L(f, P) \leq I(f, P, X) \leq U(f, P).$$

In particular, we always have $L(f, P) \leq U(f, P)$. In fact, a stronger statement holds true:

Lemma 2.4. *Let* f *be a bounded function on a closed interval* $[a, b]$. *Then, for any partitions* P *and* Q *of* $[a, b]$, *we have*

$$L(f, P) \leq U(f, Q).$$

The proof of this lemma is based on the notion of a *refinement* of a partition:

Definition 2.5. A partition R of $[a, b]$ is a *refinement* of a partition P of $[a, b]$ if R is obtained from P by adding a certain number of points, i.e., if $P \subset R$.

Lemma 2.6. *Let f be a bounded function on a closed interval $[a, b]$. Let also P be a partition of $[a, b]$, and let R be a refinement of P. Then*

$$U(f, R) \leq U(f, P), \quad L(f, R) \geq L(f, P).$$

Exercise 2.7. *Prove Lemma 2.6.*

Proposition 2.8. *Let P and Q be partitions of $[a, b]$. Then there is a partition R of $[a, b]$ which is a refinement of both P and Q.*

Proof. One can take $R = P \cup Q$. □

Now we prove Lemma 2.4.

Proof of Lemma 2.4. Let R be a common refinement of P and Q. Then, by Lemma 2.6,

$$L(f, P) \leq L(f, R).$$

Furthermore, by Proposition 2.3, we have

$$L(f, R) \leq U(f, R).$$

Finally, by Lemma 2.6, we have

$$U(f, R) \leq U(f, Q).$$

Combining these three inequalities, we get the result of the lemma. □

Now, for a bounded function f on $[a, b]$, let

$$\mathcal{U}(f) = \{U(f, P) \mid P \text{ is a partition of } [a, b]\}$$

be the set of all possible upper sums for this function. Similarly, let

$$\mathcal{L}(f) = \{L(f, P) \mid P \text{ is a partition of } [a, b]\}$$

be the set of all possible lower sums[2]. Then we have the following corollary of Lemma 2.4:

Corollary 2.9. *The set $\mathcal{U}(f)$ is bounded below, while the set $\mathcal{L}(f)$ is bounded above.*

Proof. Let P be any partition of $[a, b]$. Then, by Lemma 2.4, $L(f, P)$ is a lower bound for $\mathcal{U}(f)$, while $U(f, P)$ is an upper bound for $\mathcal{L}(f)$. □

We also get the following:

Corollary 2.10. $\sup \mathcal{L}(f) \leq \inf \mathcal{U}(f)$.

[2]Note that the sets $\mathcal{U}(f)$ and $\mathcal{L}(f)$ depend both on the function f and the interval $[a, b]$. We omit the dependence on $[a, b]$ in the notation for the sake of simplicity. This should not cause any confusion since the interval $[a, b]$ is fixed throughout the whole section.

Proof. By Lemma 2.4 we have that $L(f,P)$ is a lower bound for $\mathcal{U}(f)$ for any partition P. So, $L(f,P) \leq \inf \mathcal{U}(f)$ for any partition P, meaning that $\inf \mathcal{U}(f)$ is an upper bound for $\mathcal{L}(f)$, and thus $\inf \mathcal{U}(f) \geq \sup \mathcal{L}(f)$. □

Our further strategy is to show that $\sup \mathcal{L}(f) = \inf \mathcal{U}(f)$ for a continuous function f, and that both numbers are equal to the Riemann integral of f (in particular, f is integrable). The following lemma is the main ingredient of the proof:

Lemma 2.11. *Let f be continuous on $[a,b]$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any partition P of $[a,b]$ with* $\mathrm{mesh}(P) < \delta$ *we have $U(f,P) - L(f,P) < \varepsilon$.*

Proof. We have

$$U(f,P) - L(f,P) = \sum_{j=1}^{n} (\sup_{x\in I_j} f(x) - \inf_{x\in I_j} f(x)) \cdot \Delta_j.$$

Since f is continuous on I_j, and I_j is a closed interval, it follows that f attains its supremum and infimum on I_j, and

$$U(f,P) - L(f,P) = \sum_{j=1}^{n} (f(x_j^{max}) - f(x_j^{min})) \cdot \Delta_j$$

for certain $x_j^{max}, x_j^{min} \in I_j$. Further, since f is continuous on $[a,b]$, it follows that f is uniformly continuous on $[a,b]$ and there exists δ such that $|f(y) - f(x)| < \varepsilon/(b-a)$ for any $x, y \in [a,b]$ with $|x - y| < \delta$. In particular, if $\mathrm{mesh}(P) < \delta$, then $|x_j^{max} - x_j^{min}| < \delta$, so $f(x_j^{max}) - f(x_j^{min}) < \varepsilon/(b-a)$ (we omit the absolute value sign in the left-hand side since $f(x_j^{max}) - f(x_j^{min}) > 0$ by construction). So,

$$U(f,P) - L(f,P) = \sum_{j=1}^{n} (f(x_j^{max}) - f(x_j^{min})) \cdot \Delta_j < \sum_{j=1}^{n} \frac{\varepsilon}{b-a} \cdot \Delta_j = \varepsilon$$

whenever $\mathrm{mesh}(P) < \delta$, as desired. (Here we use that $\sum_{j=1}^{n} \Delta_j = b - a$.) □

Corollary 2.12. *Let f be continuous on $[a,b]$. Then* $\sup \mathcal{L}(f) = \inf \mathcal{U}(f)$.

Proof. Take any $\varepsilon > 0$. Using Lemma 2.11, we find a partition P of $[a,b]$ with $U(f,P) - I(f,P) < \varepsilon$. Then, since $\inf \mathcal{U}(f) \leq U(f,P)$, and $\sup \mathcal{L}(f) \geq L(f,P)$, we have

$$\inf \mathcal{U}(f) - \sup \mathcal{L}(f) \leq U(f,P) - L(f,P) < \varepsilon.$$

So, $\inf \mathcal{U}(f) - \sup \mathcal{L}(f) < \varepsilon$ for any $\varepsilon > 0$, meaning that $\inf \mathcal{U}(f) - \sup \mathcal{L}(f) \leq 0$. Combining this with Corollary 2.10, we get the result. □

Finally, we prove the main result.

Proof of Theorem 2.1. We will show that the number I defined by $I = \sup \mathcal{L}(f) = \inf \mathcal{U}(f)$ (the latter two numbers are equal by Corollary 2.12) is the integral of f. To that end, we need to prove that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any partition P of $[a,b]$ with $\mathrm{mesh}(P) < \delta$ and any evaluation sequence X we have $|I(f,P,X) - I| < \varepsilon$. Take any $\varepsilon > 0$. Then, by Lemma 2.11 there exists $\delta > 0$ such that for any partition P of $[a,b]$ with mesh$(P) < \delta$ we have $U(f,P) - L(f,P) < \varepsilon$. We show that this is δ we are looking for. Indeed, let P be any partition with mesh$(P) < \delta$, and let X be any evaluation sequence for P. Then

$$I - \varepsilon < I = \inf \mathcal{U}(f) \leq U(f,P) < L(f,P) + \varepsilon \leq \sup \mathcal{L}(f) + \varepsilon = I + \varepsilon.$$

(The inequality $U(f,P) < L(f,P) + \varepsilon$ follows from $U(f,P) - L(f,P) < \varepsilon$. The latter is true because mesh$(P) < \delta$.). Similarly,

$$I - \varepsilon = \inf \mathcal{U}(f) - \varepsilon \leq U(f,P) - \varepsilon < L(f,P) \leq \sup \mathcal{L}(f) = I < I + \varepsilon.$$

So, both $U(f,P)$ and $L(f,P)$ are in the ε-neighborhood of I. Furthermore, using Proposition 2.3, we get

$$I - \varepsilon < L(f,P) \leq I(f,P,X) \leq U(f,P) < I + \varepsilon,$$

so the Riemann sum $I(f,P,X)$ is also in the ε-neighborhood of I, as desired. □