MAT337H1, Introduction to Real Analysis: recommended problems for Jan 18 class

1. For a positive real number $x = x_0.x_1\dots$, let $[x]_n = x_0.x_1\dots x_n$. For two positive real numbers x and y, we define their sum by

$$x + y = \sup\{[x]_n + [y]_n \mid n \in \mathbb{Z}, n \geq 0\}.$$

   Show that for any three positive real numbers x, y, z, we have

$$(x + y) + z = x + (y + z).$$

   (You can use that addition of rational numbers has this property.)

2. Show that the following definition of $x + y$ is equivalent to the above:

$$x + y = \sup\{[x]_n + [y]_m \mid m, n \in \mathbb{Z}, m, n \geq 0\}.$$

3. Prove that $\inf\left\{\dfrac{1}{2n^3 - n} \mid n \in \mathbb{Z}, n > 0\right\} = 0$.

4. Let $z = \sup\{x \in \mathbb{R} \mid x > 0, x^2 < 2\}$. In class we showed that z^2 cannot be greater than 2. Show that z^2 cannot be less than 2 either. Conclude that $z^2 = 2$. (From now on we will denote this number z by $\sqrt{2}$).

5. Show that $\sup\{x \in \mathbb{Q} \mid x > 0, x^2 < 2\} = \sqrt{2}$.