MAT337H1, Introduction to Real Analysis: Solutions to Problems 1 and 2 for Jan 18 class

Problem 1. For a positive real number $x = x_0.x_1\dots$, let $[x]_n = x_0.x_1\dots x_n$. For two positive real numbers x and y, we define their sum by

$$x + y = \sup\{[x]_n + [y]_n \mid n \in \mathbb{Z}, n \geq 0\}.$$

Show that for any three positive real numbers x, y, z, we have

$$(x + y) + z = x + (y + z).$$

(You can use that addition of rational numbers has this property.)

Remark. Note that the definition of $[x]_n$ is ambiguous if x has two different decimal representations. However, one can show that $x+y$, as defined above, does not depend on the choice of decimal representations of x and y. Another way to overcome this problem is to prohibit decimal representations of the form $x_0.x_1\dots x_n999\dots$, and we take this approach here.

Solution of Problem 1. Let $A = \{[x+y]_n + [z]_n \mid n \in \mathbb{Z}, n \geq 0\}$, $B = \{[x]_n + [y+z]_n \mid n \in \mathbb{Z}, n \geq 0\}$. We need to show that $\sup A = \sup B$. To prove this, we introduce the set $C = \{[x]_n + [y]_n + [z]_n \mid n \in \mathbb{Z}, n \geq 0\}$ and show that $\sup A = \sup C$ and $\sup B = \sup C$. It suffices to show that $\sup A = \sup C$. The proof of $\sup B = \sup C$ is then achieved by swapping x and z.

Let $a_n = [x+y]_n + [z]_n$, $c_n = [x]_n + [y]_n + [z]_n$. Notice that $a_n \geq c_n$. Indeed, by definition of $x + y$ we have $x + y \geq [x]_n + [y]_n$. Therefore, $[x + y]_n \geq [[x]_n + [y]_n]_n$ (this follows from the definition of comparison of real numbers; it is important that we do not use decimal representations of the form $x_0.x_1\dots x_n999\dots$). But the number $[x]_n + [y]_n$ has at most n digits after the decimal point. Therefore, $[[x]_n + [y]_n]_n = [x]_n + [y]_n$, proving that $a_n \geq c_n$.

Now, since $A = \{a_n \mid n \in \mathbb{Z}, n \geq 0\}$, $C = \{c_n \mid n \in \mathbb{Z}, n \geq 0\}$, and $a_n \geq c_n$, it follows that $\sup A \geq \sup C$ (check this).

Further, it is easy to see that for any positive real number w and any positive integers m, n, one has $[w]_m \leq [w]_n + 10^{-n}$. Therefore, $[x]_m + [y]_m \leq [x]_n + [y]_n + 2 \cdot 10^{-n}$, and it follows that for $x + y$, defined as the supremum of the left-hand side over all m's, we have $x + y \leq [x]_n + [y]_n + 2 \cdot 10^{-n}$ for any positive integer n. Thus, $[x + y]_n \leq [x]_n + [y]_n + 2 \cdot 10^{-n}$, i.e., $a_n \leq c_n + 2 \cdot 10^{-n}$.

So, since $A = \{a_n \mid n \in \mathbb{Z}, n \geq 0\}$, $C = \{c_n \mid n \in \mathbb{Z}, n \geq 0\}$, and $a_n \leq c_n + 2 \cdot 10^{-n}$, it follows that $\sup A \leq \sup C$ (check this). Now we know that $\sup A \geq \sup C$ and $\sup A \leq \sup C$. So, $\sup A = \sup C$, as desired.

Problem 2. Show that the following definition of $x + y$ is equivalent to the above:

$$x + y = \sup\{[x]_n + [y]_m \mid m, n \in \mathbb{Z}, m, n \geq 0\}.$$

Solution: Let $A = \{[x]_n + [y]_n \mid n \in \mathbb{Z}, n \geq 0\}$, $B = \{[x]_n + [y]_m \mid m, n \in \mathbb{Z}, m, n \geq 0\}$. We need to show that $\sup A = \sup B$ (both sets are non-empty and bounded above, e.g., by $[x]_0 + [y]_0 + 2$, so these supremums are well-defined.) We use the following lemma.

Lemma. Let $A, B \subset \mathbb{R}$ be non-empty bounded above sets satysfying the following conditions:

1. For any $a \in A$, there exists $b \in B$ such that $b \geq a$.

2. For any $b \in B$, there exists $a \in A$, such that $a \geq b$.

Then $\sup A = \sup B$.

Proof of the lemma.

▶ Take any $a \in A$. Then there exists $b \in B$ such that $b \geq a$. But since $b \leq \sup B$, it follows that $a \leq \sup B$. So $\sup B$ is an upper bound for A, and it follows that $\sup B \geq \sup A$. Analogously, one proves that $\sup A$ is an upper bound for B, so $\sup B \geq \sup A$. We conclude that $\sup A = \sup B$, as desired. ◀

Now, it suffices to show that our sets A and B satisfy the conditions of the lemma. The first condition (for any $a \in A$, there exists $b \in B$ such that $b \geq a$) is obvious, since $A \subset B$, and we can take $b = a$. So, we only need to verify the second condition (for any $b \in B$, there exists $a \in A$, such that $a \geq b$). Take any $b \in B$. Then $b = [x]_n + [y]_m$, where m, n are non-negative integers. Let $k = \max(m, n)$. Then we have $[x]_k \geq [x]_n$ and $[y]_k \geq [y]_m$, so $[x]_k + [y]_k \geq b$. Since, $[x]_k + [y]_k \in A$, this ends the proof.