MAT337H1, Introduction to Real Analysis: Solutions to Exercise J for Section 2.4 and Exercises C and J for Section 2.5

Exercise J for Section 2.4. Let a_0, a_1 be positive real numbers, and set

$$a_{n+2} = \sqrt{a_{n+1}} + \sqrt{a_n}$$

for $n \geq 0$.

(a) Show that there is N such that $a_n \geq 1$ for all $n \geq N$.

(b) Let $\varepsilon_n = |a_n - 4|$. Show that $\varepsilon_{n+2} \leq \frac{1}{3}(\varepsilon_{n+1} + \varepsilon_n)$ for $n \geq N$.

(c) Prove that the sequence a_n converges.

Solution. (a) We first show that there exists n such that $a_n \geq 1$. Assume, for the sake of contradiction, that $a_n < 1$ for any n. Then the sequence a_n is bounded above. Furthermore, we have

$$a_{n+2} = \sqrt{a_{n+1}} + \sqrt{a_n} \geq \sqrt{a_{n+1}} > a_{n+1},$$

meaning that the sequence $a_2, a_3, \ldots$ is increasing. So, this sequence converges by the monotone converges theorem. Let a be its limit. Then for any $\varepsilon > 0$ there exists k such that $a_n > a - \varepsilon$ for any $n \geq k$. In particular, we have $a_k > a - \varepsilon$, $a_{k+1} > a - \varepsilon$. Therefore,

$$a_{k+2} = \sqrt{a_{k+1}} + \sqrt{a_k} > 2\sqrt{a - \varepsilon}.$$

Further, notice that since $a_n < 1$ for every n, we have $a \leq 1$ (see, e.g., Exercise C for Section 2.4). So,

$$a_{k+2} > 2\sqrt{a - \varepsilon} > 2(a - \varepsilon).$$

Choosing $\varepsilon = \frac{1}{2}a$ (we can take such ε because $a > 0$), we get

$$a_{k+2} > 2(a - \varepsilon) = a.$$

But this is impossible, since a_n is an increasing sequence, which implies

$$\sup\{a_n\} = \lim_{n\to\infty} a_n = a.$$

So, our assumption is wrong, and there exists n such that $a_n \geq 1$. But then

$$a_{n+2} = \sqrt{a_{n+1}} + \sqrt{a_n} \geq \sqrt{a_n} \geq 1.$$

Proceeding by induction, one can show that $a_k \geq 1$ for $k \geq n+2$. So, one can take $N = n+2$.

(b) For $n \geq N$, we have

$$\varepsilon_{n+2} = |a_{n+2} - 4| = |\sqrt{a_{n+1}} + \sqrt{a_n} - 4| = |(\sqrt{a_{n+1}} - 2) + (\sqrt{a_n} - 2)| \leq |\sqrt{a_{n+1}} - 2|$$
$$+|\sqrt{a_n} - 2| = \frac{|a_{n+1} - 4|}{\sqrt{a_{n+1}} + 2} + \frac{|a_n - 4|}{\sqrt{a_n} + 2} \leq \frac{|a_{n+1} - 4|}{3} + \frac{|a_n - 4|}{3} = \frac{1}{3}(\varepsilon_{n+1} + \varepsilon_n),$$

where we used that $a_n, a_{n+1} \geq 1$ and hence $\sqrt{a_n} + 2, \sqrt{a_{n+1}} + 2 \geq 3$.

(c) We first show that $\lim_{n\to\infty} \varepsilon_n = 0$. Let $\delta_1 = \max(\varepsilon_1, \varepsilon_2)$, $\delta_2 = \max(\varepsilon_3, \varepsilon_4)$, etc. In general, we have $\delta_n = \max(\varepsilon_{2n-1}, \varepsilon_{2n})$. Then, provided that $2n - 1 \geq N$, we have

$$\varepsilon_{2n+1} \leq \frac{1}{3}(\varepsilon_{2n} + \varepsilon_{2n-1}) \leq \frac{1}{3}(\max(\varepsilon_{2n}, \varepsilon_{2n-1}) + \max(\varepsilon_{2n}, \varepsilon_{2n-1})) = \frac{2}{3}\max(\varepsilon_{2n}, \varepsilon_{2n-1}) = \frac{2}{3}\delta_n,$$

and

$$\varepsilon_{2n+2} \leq \frac{1}{3}(\varepsilon_{2n+1} + \varepsilon_{2n}) \leq \frac{1}{3}\left(\frac{2}{3}\delta_n + \max(\varepsilon_{2n}, \varepsilon_{2n-1})\right) = \frac{5}{9}\delta_n \leq \frac{2}{3}\delta_n.$$

So, for $2n - 1 \geq N$, we have

$$\delta_{n+1} = \max(\varepsilon_{2n+1}, \varepsilon_{2n+2}) \leq \frac{2}{3}\delta_n.$$

From the latter it follows that $\lim_{n\to\infty} \delta_n = 0$ (check this). So, the limit of the sequence $\delta_1, \delta_1, \delta_2, \delta_2, \delta_3, \delta_3, \ldots$ is also 0. At the same time, the terms of this sequence estimate the terms of the sequence ε_n from above. So, $\lim_{n\to\infty} \varepsilon_n = 0$ by the squeeze theorem (where we also use that $\varepsilon_n \geq 0$). Further,

$$-\varepsilon_n \leq a_n - 4 \leq \varepsilon_n,$$

so $\lim_{n\to\infty}(a_n - 4) = 0$ also by the squeeze theorem. Thus, $\lim_{n\to\infty} a_n = 4$.

Exercise C for Section 2.5. If $\lim_{n\to\infty} a_n = L > 0$, prove that $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$.

Solution. We are given that $\lim_{n\to\infty} a_n = L > 0$. We first show that there exists $N_0 \in \mathbb{Z}$ such that if $n \in \mathbb{Z}$ and $n \geq M$, then $a_n > 0$. By definition of limit, we have that for any $\varepsilon > 0$ there exists $N(\varepsilon)$ such that

$$|a_n - L| < \varepsilon \text{ for } n \geq N(\varepsilon). \tag{1}$$

Applying this for $\varepsilon = L$, we get that for any $n \geq N(L)$ one has $|a_n - L| < L$. The latter, in particular, means that $a_n > 0$. So, $N(L)$ is the desired N_0.

Since $a_n > 0$ for n sufficiently large, $\sqrt{a_n}$ makes sense. Although the first few terms of the latter sequence may be undefined, it still makes sense to discuss its limit.

Now we show that $\sqrt{a_n} \to \sqrt{L}$. For this we need to show that for any $\delta > 0$ there exists $N'(\delta)$ such that

$$|\sqrt{a_n} - \sqrt{L}| < \varepsilon \text{ for } n \geq N'(\delta). \tag{2}$$

(Here one should have $N'(\delta) \geq N_0$, so that $\sqrt{a_n}$ is well-defined.) We fix $\delta > 0$ and find the number $N'(\delta)$. For any $n \geq N_0$, we have

$$|\sqrt{a_n} - \sqrt{L}| = \frac{|a_n - L|}{\sqrt{a_n} + \sqrt{L}} \leq \frac{|a_n - L|}{\sqrt{L}}.$$

Further, applying (1) for $\varepsilon = \delta\sqrt{L}$, we get that $|a_n - L| < \delta\sqrt{L}$ for $n \geq N(\delta\sqrt{L})$. So, for such numbers n, we have

$$|\sqrt{a_n} - \sqrt{L}| \leq \frac{|a_n - L|}{\sqrt{L}} < \delta,$$

provided that $\sqrt{a_n}$ is well-defined. This means that as $N'(\delta)$ one can take any number N such that $N \geq N(\delta\sqrt{L})$ and $N \geq N_0$. For example, one can take $N'(\delta) = \max(N(\delta\sqrt{L}), N_0)$. So, for any $\delta > 0$ we found $N'(\delta)$ satisfying (2), meaning that $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$.

Exercise J for Section 2.5. Show that the set $S = \{n + m\sqrt{2} \mid m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.

Solution. Let $S_+ = \{x \in S \mid x > 0\}$. Since the set S_+ is non-empty and bounded below (by 0), it has a greatest lower bound.

Lemma. $\inf S_+ = 0$.

► Let $a = \inf S_+$. Since 0 is a lower bound for S_+, we have $a \geq 0$. Assume that $a > 0$. Then there are two possible cases: either $a \notin S_+$, or $a \in S_+$. In the first case, since a is the greatest lower bound, we have that for any $b > a$ there exists $x \in S_+$ such that $x < b$ (otherwise b would be a lower bound for S_+ greater than a). Taking $b = 2a$, we find $x_1 \in S_+$ such that $x_1 < 2a$. Further, taking $b = x_1$ (notice that $x_1 > a$ since $a \notin S_+$), we find $x_2 \in S_+$ such that $x_2 < x_1$. Then

$$a < x_2 < x_1 < 2a.$$

Notice that since $x_1, x_2 \in S$, we have $x_2 - x_1 \in S$. (This follows from the definition of S: since $x_1, x_2 \in S$, we have $x_1 = n_1 + m_1\sqrt{2}$, $x_2 = n_2 + m_2\sqrt{2}$, where $m_1, n_1, m_2, n_2 \in \mathbb{Z}$, so $x_2 - x_1 = (n_1 + m_1\sqrt{2}) - (n_2 + m_2\sqrt{2}) = (n_2 - n_1) + (m_2 - m_1)\sqrt{2} \in S$). Furthermore, since $x_2 > x_1$, we in fact have $x_2 - x_1 \in S_+$. But $x_2 - x_1 < a$, showing that a is not a lower bound, which is a contradiction.

Now we consider the case $a \in S_+$. In this case, it follows from the definition of S that $na \in S$ for any $n \in \mathbb{Z}$. Moreover, there are no other elements in S. Indeed if $x \in S$ is not of the form na, where $n \in \mathbb{Z}$, then it lies in the certain interval of the form $(na, (n+1)a)$, where $n \in \mathbb{Z}$. But then $x - na < a$, and $x - na \in S_+$, which contradicts a being a lower bound for S_+. So, we must have $S = \{na \mid n \in \mathbb{Z}\}$. This, in particular, means that $1 = na$ and $\sqrt{2} = ma$ for certain integers m, n. But then

$$\sqrt{2} = \frac{\sqrt{2}}{1} = \frac{ma}{na} = \frac{m}{n},$$

which is impossible, since $\sqrt{2}$ is irrational. So, our assumption $a > 0$ is false, meaning that $a = 0$. ◄

Now we show that S is dense in $\mathbb{R}$. Take any $x \in \mathbb{R}$. We need to construct a sequence $x_n \in S$ such that $\lim_{n\to\infty} x_n = x$. It follows from the Lemma that for any integer $n > 0$ we can find $a_n \in S_+$ such that $a_n < \frac{1}{n}$. Consider the set of numbers of the form $ma_n, m \in \mathbb{Z}$. These numbers partition the real line into intervals of length a_n, which is less than $\frac{1}{n}$. The number x lies in one of these intervals. This means that there exists $m \in \mathbb{Z}$ such that

$$ma_n \leq x \leq (m+1)a_n.$$

In particular, we have

$$|x - x_n| \leq |(m+1)a_n - ma_n| = a_n.$$

Set $x_n = ma_n$. Doing this for every positive integer n, we obtain a sequence x_n. By construction, we have $x_n \in S$. (Here we use that $a_n \in S \Rightarrow ma_n \in S$ for any $m \in \mathbb{Z}$.) Furthermore, we have $|x - x_n| \leq a_n$, also by construction. So, $|x - x_n| < \frac{1}{n}$. But this imples $\lim_{n\to\infty} x_n = x$ (e.g., by the squeeze theorem), as desired.