MAT337H1, Introduction to Real Analysis: additional recommended problems for Jan 27 class

1. Suppose that $I_n = (a_n, b_n) = \{x \in \mathbb{R} \mid a_n < x < b_n\}$ are non-empty *open* intervals such that $I_{n+1} \subseteq I_n$ for every $n \geq 1$ (i.e. I_{n+1} is a subset of I_n, possibly coinciding with I_n itself). Is it true that the intersection $\bigcap_{n\geq 1} I_n$ must be non-empty? Is it true that finite intersections $\bigcap_{n=1}^{N} I_n$ are non-empty?

   Remark: In mathematical literature, $A \subset B$ does not always mean strict inclusion, i.e. one can write that $A \subset B$ even if $A = B$. So, the meaning of $A \subset B$ and $A \subseteq B$ is often the same. To emphasize that $A \subset B$ but $A \neq B$, one can write $A \subsetneq B$ or say that A is a *proper* subset of B.

   Since $A \subset B$ means different things in different sources, one should be careful when using this notation.

2. Suppose that $I_n = [a_n, b_n] = \{x \in \mathbb{R} \mid a_n \leq x \leq b_n\}$ are non-empty closed intervals such that $I_{n+1} \subseteq I_n$ for every $n \geq 1$. Let also l_n be the length of the interval I_n, i.e. $l_n = b_n - a_n$. Prove the following.

   (a) The sequence l_n converges.

   (b) If $\lim_{n\to\infty} l_n = 0$, then the set $\bigcap_{n\geq 1} I_n$ consists of one element.

   (c) If $\lim_{n\to\infty} l_n \neq 0$, then the set $\bigcap_{n\geq 1} I_n$ is infinite.