MAT337H1, Introduction to Real Analysis: Solution of Exercise D for Section 2.7 and Question 2 from the recommended problems PDF for Jan 27

Exercises D. Show that every sequence has a monotone subsequence.

Solution. Let a_n be a sequence (of real numbers). We need to show that it has a monotone subsequence. First assume that the sequence a_n is bounded. Then, by Bolzano-Weierstrass theorem, a_n has a convergent subsequence, which we call b_n. Now we show that b_n has a monotone subsequence. Let L be the limit of b_n. Notice that if b_n has infinitely many terms equal to L, then we are done: $L, L, \dots$ is a monotone subsequence of a_n. So, we may assume that only finitely many terms of b_n are equal to L. Then either there are infinitely many terms satisfying $b_n > L$, or there are infinitely many terms satisfying $b_n < L$ (maybe both). We consider the first case. The second one is completely analogous. Take any term of b_n greater than L and call it c_1. Take $\varepsilon = (c_1 - L)/2$. Then, since $b_n \to L$, there are only finitely many terms of b_n outside the ε-neighborhood of L. At the same time, there are infinitely many terms satisfying $b_n > L$. So, there must be infinitely many terms of b_n between L and $L + \varepsilon$. Take one of them and call it c_2. Notice that $c_2 < L + \varepsilon < c_1$. Then take $\varepsilon = (c_2 - L)/2$ and repeat the procedure. This gives $c_3 < c_2$. Continuing this, we get a decreasing subsequence of the sequence a_n, as desired.

Now, assume that a_n is not bounded. Then it must be unbounded either from above, or from below (maybe both). We consider the case when it is unbounded above. The other case is analogous. Take any term of the sequence a_n and call it c_1. Then there is another term of a_n which is larger than c_1: otherwise, c_1 would be an upper bound for a_n. Take this larger term and call it c_2. Continuing this procedure, we get an increasing subsequence of a_n, as desired.

Question 2 from the PDF file. Suppose that $I_n = [a_n, b_n] = \{x \in \mathbb{R} \mid a_n \leq x \leq b_n\}$ are non-empty closed intervals such that $I_{n+1} \subseteq I_n$ for every $n \geq 1$. Let also l_n be the length of the interval I_n, i.e. $l_n = b_n - a_n$. Prove the following.

(a) The sequence l_n converges.

(b) If $\lim_{n\to\infty} l_n = 0$, then the set $\bigcap_{n\geq 1} I_n$ consists of one element.

(c) If $\lim_{n\to\infty} l_n \neq 0$, then the set $\bigcap_{n\geq 1} I_n$ is infinite.

Solution.

(a) Since $I_{n+1} \subseteq I_n$, we have $l_{n+1} \leq l_n$. So, the sequence l_n is non-increasing and bounded below (by 0). Therefore, it is convergent (by the monotone convergence theorem).

(b) The set $\bigcap_{n\geq 1} I_n$ is non-empty by the nested intervals lemma. So, to show that it consists of one element, it suffices to prove that it cannot consist of more than one element. Assume the contrary, i.e. that there are $x, y \in \bigcap_{n\geq 1} I_n$ such that $x \neq y$. Then, by definition of intersection, we have $x, y \in I_n$ for any integer $n \geq 1$. But this implies $l_n \geq |x - y| > 0$, which is not possible since $\lim_{n\to\infty} l_n = 0$. So, our assumption is false, and the set $\bigcap_{n\geq 1} I_n$ consists of exactly one element.

(c) The sequence a_n is non-decreasing and bounded above (by b_1), so it is convergent. Analogously, b_n is non-increasing and bounded below, so it is convergent as well. Furthermore, we have

$$\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = \lim_{n\to\infty} l_n > 0. \tag{1}$$

Also notice that since a_n is non-increasing, we have $\lim_{n\to\infty} a_n = \sup\{a_n\}$. (See the proof of the monotone convergence theorem.) Analogously, $\lim_{n\to\infty} b_n = \inf\{b_n\}$. So, by (1), we have $\inf\{b_n\} > \sup\{a_n\}$. Now take any real number x such that

$$\inf\{b_n\} \geq x \geq \sup\{a_n\}.$$

(There are infinitely many such real numbers.) Then, since $x \geq \sup\{a_n\}$, we have that $x \geq a_n$ for any positive integer n. Analogously, $x \leq b_n$. So, $x \in I_n$ for every n, which means that $x \in \bigcap_{n\geq 1} I_n$. So, the latter intersection contains infinitely many elements. (In fact, one has $\bigcap_{n\geq 1} I_n = [\sup\{a_n\}, \inf\{b_n\}]$).