MAT337H1, Introduction to Real Analysis: additional recommended problems for Mar 1 class

1. Prove the following fact which we used in the proof of Fermat's theorem. Let f be a function defined in all points of an interval (a, b) except, possibly, one point $x_0 \in (a, b)$. Assume also that f changes sign at x_0. Further, assume that there exists a limit $\lim_{x \to x_0} f(x)$. Then $\lim_{x \to x_0} f(x) = 0$.

2. Let $p(x)$ be a polynomial of degree n. Assume that $p(x)$ has n real roots, counting with multiplicities. Prove that the polynomial $p'(x)$ has $n-1$ real roots, counting with multiplicities.

3. Let $p(x) = ax^3 + bx^2 + cx + d$ be a polynomial of degree 3 with leading coefficient $a > 0$. Show that the following conditions are equivalent:

   (a) p has three distinct real roots;

   (b) p' has two distinct real roots $x_1 < x_2$ that satisfy $p(x_1) > 0$ and $p(x_2) < 0$.

   Hence determine the number of real roots of the polynomial $x^3 - x + 1$.