MAT337H1, Introduction to Real Analysis: recommended problems for Mar 15 class

1. Show that if f is integrable on $[a, b]$, then its integral is uniquely defined. In other words, there can exist at most one number I with the following property: For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any partition P of $[a, b]$ with $\mathrm{mesh}(P) < \delta$ and any evaluation sequence X for P we have $|I(f, P, X) - I| < \varepsilon$.

2. Let f be a function bounded on $[a, b]$. Let also P be a partition of $[a, b]$, and let R be a refinement of P (i.e. $R \supset P$). Show that the corresponding upper sums satisfy $U(f, P) \geq U(f, R)$, while the lower sums satisfy $L(f, P) \leq L(f, R)$.

3. Prove the following statement which we used to show that every continuous function is integrable: Let $A, B \subset \mathbb{R}$ be non-empty sets such that for any $a \in A$ and any $b \in B$ we have $a \leq b$. Then A is bounded above, B is bounded below, and $\sup A \leq \inf B$.

4. Let f be a function integrable on $[a, b]$. Let also P_n be a sequence of partitions of $[a, b]$ such that $\lim_{n\to\infty} \mathrm{mesh}(P_n) = 0$, and let X_n be an evaluation sequence for P_n. Prove that
$$\lim_{n\to\infty} I(f, P_n, X_n) = \int_a^b f(x)dx.$$

5. Let f be integrable on $[a, b]$. Show that f is bounded on $[a, b]$.

6. Show that a bounded function f is integrable on $[a, b]$ if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any partition P of $[a, b]$ with $\mathrm{mesh}(P) < \delta$ we have $U(f, P) - L(f, P) < \varepsilon$.

7. (This is a difficult problem.) Let f be a bounded function on $[a, b]$ continuous at all points except finitely many. Prove that f is integrable on $[a, b]$.