MAT337H1, Introduction to Real Analysis: solution to recommended problem 6 for Mar 15 class

Problem. Show that for a bounded function f on $[a, b]$ the following conditions are equivalent:

(a) f is integrable on $[a, b]$;

(b) for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any partition P of $[a, b]$ with $\text{mesh}(P) < \delta$ we have $U(f, P) - L(f, P) < \varepsilon$.

Solution. The proof of the implication (b) $\Rightarrow$ (a) follows the lines of that of Theorem 2.1 from the lecture notes. Indeed, in that theorem one assumes that f is continuous, but continuity is only used in the proof of Lemma 2.1. Since Lemma 2.1 holds in our case by assumption (b), the rest of the proof works without any modifications.

The proof of the implication (a) $\Rightarrow$ (b) is based on the following observation:

Lemma. For any partition P of $[a, b]$ one has

$$U(f, P) = \sup\{I(f, P, X)\}, \quad L(f, P) = \inf\{I(f, P, X)\},$$

where $\{I(f, P, X)\}$ stands for the set of Riemannian sums obtained by taking all possible evaluation sequences X for P.

Proof of the lemma. ► We prove that $U(f, P) = \sup\{I(f, P, X)\}$. The proof of the second equality is analogous. First notice that $U(f, P)$ is an upper bound for $\{I(f, P, X)\}$ by Proposition 2.3 from the lecture notes. So, to prove that $U(f, P)$ is the least upper bound it suffices to show that for any $\varepsilon > 0$ there exists an evaluation sequence X for P such that $I(f, P, X) > U(f, P) - \varepsilon$. Let $P = \{x_0 < x_1 < \cdots < x_{n-1} < x_n\}$, and let $I_j = [x_{j-1}, x_j]$. Then we can find points $x_j^* \in I_j$ such that $f(x_j^*) > (\sup_{x \in I_j} f(x)) - \frac{\varepsilon}{b-a}$. (Otherwise the number on the right-hand side is an upper bound for $f(x)$ on the interval I_j, which contradicts the definition of the supremum $\sup_{x \in I_j} f(x)$).) Taking $\{x_1^*, \ldots, x_n^*\}$ as an evaluation sequence, we get $I(f, P, X) > U(f, P) - \varepsilon$, as desired.◄

Now we prove the implication (a) $\Rightarrow$ (b). Take any $\varepsilon > 0$. Then, since f is integrable, there exists $\delta > 0$ such that $|I(f, P, X) - \int_a^b f(x)dx| < \frac{\varepsilon}{3}$ for any partition P with $\text{mesh}(P) < \delta$ and any evaluation sequence X. This can be rewritten as

$$\int_a^b f(x)dx - \frac{\varepsilon}{3} < I(f, P, X) < \int_a^b f(x)dx + \frac{\varepsilon}{3}.$$

So, for any partition P with $\text{mesh}(P) < \delta$ the number $\int_a^b f(x)dx + \frac{\varepsilon}{3}$ is an upper bound for the set of Riemannian$\{I(f, P, X)\}$, and it follows that

$$U(f, P) = \sup\{I(f, P, X)\} \leq \int_a^b f(x)dx + \frac{\varepsilon}{3}.$$

Similarly, we get

$$L(f, P) = \inf\{L(f, P, X)\} \geq \int_a^b f(x)dx - \frac{\varepsilon}{3}.$$

So, $U(f, P) - L(f, P) \leq \frac{2\varepsilon}{3} < \varepsilon$, as desired.