MAT337H1, Introduction to Real Analysis: solution of Problem 2 for Mar 17 class

Problem. For $x > 0$, define a function $\ln(x)$ by the formula

$$\ln(x) = \int_1^x \frac{1}{t} dt.$$

Show, using this definition, that $\ln(xy) = \ln(x) + \ln(y)$. Hence show that ln is a bijection from positive numbers to all real numbers.

Solution. We have

$$\ln(xy) = \int_1^{xy} \frac{1}{t} dt \overset{\text{by Problem 1}}{=} \int_{1/y}^{x} \frac{1}{t} dt \overset{\text{by Ex. K, L, Section 6.3}}{=} \int_1^x \frac{1}{t} dt - \int_1^{1/y} \frac{1}{t} dt$$

$$\overset{\text{by Problem 1}}{=} \int_1^x \frac{1}{t} dt - \int_y^1 \frac{1}{t} dt = \int_1^x \frac{1}{t} dt + \int_1^y \frac{1}{t} dt = \ln(x) + \ln(y),$$

as desired. Now we show that $\ln\colon \mathbb{R}_+ \to \mathbb{R}$ (where $\mathbb{R}_+$ is the set of positive numbers) is a bijection. First notice that ln is an increasing function. Indeed, if $y > x > 0$, then

$$\ln(y) - \ln(x) = \int_1^y \frac{1}{t} dt - \int_1^x \frac{1}{t} dt \overset{\text{by Ex. K, L, Section 6.3}}{=} \int_x^y \frac{1}{t} dt$$

$$\overset{\text{by Ex. H, Section 6.3}}{\geq} \int_x^y \frac{1}{y} dt = \frac{y - x}{y} > 0.$$

This in particular means that $\ln\colon \mathbb{R}_+ \to \mathbb{R}$ is injective. To prove that ln is surjective, we need to show that the for every $y \in \mathbb{R}$ there exists $x \in \mathbb{R}_+$ such that $\ln(x) = y$. First notice that

$$\ln(1) = \int_1^1 \frac{1}{t} dt = 0.$$

Therefore, if $y = 0$, then $x = 1$ is what we are looking for. Now assume that $y > 0$. Take any number $k > 1$. Then, for any natural number n, we have

$$\ln(k^n) = \ln(k \times \cdots \times k) \overset{\text{using } \ln(xy)=\ln(x)+\ln(y)}{=} \ln(k) + \cdots + \ln(k) = n \ln(k).$$

Therefore, one can find n such that $\ln(k^n) > y$. (Any $n > y/\ln(k)$ works. Here we use that $\ln(k) > 0$, which is true because ln is increasing.) So, we have $\ln(1) < y < \ln(k^n)$. Therefore, by the intermediate value theorem there exists $x \in (1, k^n)$ such that $\ln(x) = y$, as desired. (Here we use that ln is continuous, which is true by the fundamental theorem of calculus.)

Now, let $y < 0$. Then, according to what we already proved, there exists w such that $\ln(w) = -y$. At the same time,

$$\ln(w) + \ln(1/w) = \ln(w \cdot 1/w) = \ln(1) = 0,$$

so $\ln(1/w) = y$, and $x = 1/w$ is what we are looking for.