MAT337H1, Introduction to Real Analysis: solutions to additional recommended problems for Mar 1 class

1. Prove the following fact which we used in the proof of Fermat's theorem. Let f be a function defined in all points of an interval (a, b) except, possibly, one point $x_0 \in (a, b)$. Assume also that f changes sign at x_0. Further, assume that there exists a limit $\lim_{x \to x_0} f(x)$. Then $\lim_{x \to x_0} f(x) = 0$.

   Solution. Assume that $\lim_{x \to x_0} f(x) = L \neq 0$. Taking $|L|$ as ε in the definition of the limit, we get that there is $\delta > 0$ such that for any x satisfying $0 < |x - x_0| < \delta$ we have $|f(x) - L| < |L|$ (i.e. the distance from $f(x)$ to L is less than the distance from L to 0). From the latter it follows that for x satisfying $0 < |x - x_0| < \delta$ the function $f(x)$ has the same sign as L, which contradicts $f(x)$ changing sign at x_0. So, we must have $\lim_{x \to x_0} f(x) = 0$.

2. Let $p(x)$ be a polynomial of degree n. Assume that $p(x)$ has n real roots, counting with multiplicities. Prove that the polynomial $p'(x)$ has $n - 1$ real roots, counting with multiplicities.

   Solution. Let $f(x)$ be an arbitrary polynomial having root a of multiplicity m. This means that $f(x)$ can be divided by $(x - a)$ exactly m times, i.e., $f(x) = (x - a)^m g(x)$, where $g(x)$ is a polynomial such that $g(a) \neq 0$. Differentiating, we get $f'(x) = m(x - a)^{m-1} g(x) + (x - a)^m g'(x) = (x - a)^{m-1}(mg(x) + (x - a)g'(x))$. The second factor does not vanish at a, so $f'(x)$ has root a of multiplicity $m - 1$. (In particular, if $m = 1$, then a is a root of multiplicity 0, i.e., not a root of $f'(x)$.) So, differentiation reduces multiplicity by 1.

   Now, let $x_1 < x_2 < \cdots < x_{k-1} < x_k$ be the roots of $p(x)$, and let $m_1, \ldots, m_k$ be their multiplicities. We are given that $m_1 + \cdots + m_k = n$. For the derivative $p'(x)$, these roots have multiplicities $m_1 - 1, \ldots, m_k - 1$. We have $(m_1 - 1) + \cdots + (m_k - 1) = n - k$, so we need to find $k - 1$ more roots of $p'(x)$. This is done using Rolle's theorem. From this theorem it follows that $p'(x)$ has a root in each of the $k - 1$ intervals $(x_1, x_2), \ldots, (x_{k-1}, x_k)$. So, there are $n - k$ (counting with multiplicities) roots of $p'(x)$ that are also roots of $p(x)$, and $k - 1$ roots given by Rolle's theorem. All together, we get $n - 1$ roots, as desired.

3. Let $p(x) = ax^3 + bx^2 + cx + d$ be a polynomial of degree 3 with leading coefficient $a > 0$. Show that the following conditions are equivalent:

   (a) p has three distinct real roots;

   (b) p' has two distinct real roots $x_1 < x_2$ that satisfy $p(x_1) > 0$ and $p(x_2) < 0$.

   Hence determine the number of real roots of the polynomial $x^3 - x + 1$.

   Solution. We first show that there exists $M > 0$ such that $p(x) < 0$ for $x < -M$ and $p(x) > 0$ for $x > M$. We have

$$p(x) = ax^3 + bx^2 + cx + d = x^3(a + by + cy^2 + dy^3),$$

where $y = 1/x$. The second factor (call it $f(y)$) is a continuous function of y taking a positive value when $y = 0$. So, there is $\delta > 0$ such that $f(y) > 0$ whenever $|y| < \delta$ (see Problem 4 for Feb 10 class). In other words, f (regarded as a function of $x = 1/y$) is positive when $|x| > 1/\delta$. But this means that for $|x| > 1/\delta$ the sign of $p(x)$ is the same as the sign of x^3, and one can take $1/\delta$ as M.

Now, we show that (a) implies (b). Let $a_1 < a_2 < a_3$ be the roots of $p(x)$. Then, by Rolle's theorem, $p'(x)$ has roots $x_1 \in (a_1, a_2)$ and $x_2 \in (a_2, a_3)$. We need to show that $p(x_1) > 0$ and $p(x_2) < 0$. Notice that $p(x)$ does not change sign in the intervals $(-\infty, a_1), (a_1, a_2), (a_2, a_3)$. (If it did change sign in one of these intervals, it would have a zero inside this interval by the intermediate value theorem). Furthermore, all roots of $p(x)$ have multiplicity 1 (since a polynomial of degree 3 has at most 3 roots counting with multiplicities), so $p(x)$ changes sign at each of its roots. Finally, notice that since $p(x) < 0$ for $x < -M$, we have that $p(x) < 0$ on $(-\infty, a_1)$. Therefore, $p > 0$ on (a_1, a_2) and $p < 0$ on (a_2, a_3). The result follows.

Further, we show that (b) implies (a). We have that $p(x) < 0$ for $x < -M$. In particular, $p(-M-1) < 0$. Using also that $p(x_1) > 0$, we conclude by the intermediate value theorem that p has a root in $(-M-1, x_1)$. Analogously p has a root in (x_1, x_2) and a root $(x_2, M+1)$, i.e., three distinct roots all together, as desired.

Now we apply this to $p(x) = x^3 - x + 1$. We have $p'(x) = 3x^2 - 1$. The roots of the derivative are (in increasing order) $x_1 = -1/\sqrt{3}$ and $x_2 = 1/\sqrt{3}$. We have

$$p(x_2) = \frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}} + 1 = 1 - \frac{2}{3\sqrt{3}} > 0,$$

so $p(x)$ cannot have three distinct roots. It also does not have multiple roots, since it does not vanish at the roots of its derivative. Further, we notice that a polynomial of degree 3 cannot have at exactly two distinct roots of multiplicity 1. (If it does, and these roots are, say, a and b, then dividing the polynomial by $(x-a)(x-b)$, we find the third root.) Finally, $p(x)$ should have at least one root since $p(x) < 0$ for $x < -M$ and $p(x) > 0$ for $x > M$. So, $p(x)$ has exactly one root.