MAT337H1, Introduction to Real Analysis: recommended problems for Mar 22 class

1. Prove the following inequalities, which we used in the proof of the triangle inequality for the uniform metric:

   (a) If f, g are bounded functions on a set X, and $f(x) \leq g(x)$ for any $x \in X$, then $\sup_X f \leq \sup_X g$.

   (b) If f, g are bounded functions on a set X, then $\sup_X(f+g) \leq \sup_X f + \sup_X g$.

2. Define a distance function on $\mathbb{R}^n$ by setting

$$\rho(x,y) = \sum_{i=1}^{n} |x^i - y^i|,$$

   where x^i stands for the i'th coordinate of x. Show that $\rho(x,y)$ is a metric. (This metric is called the Manhattan metric.)

3. Define a distance function on the set $C[a,b]$ of continuous functions on $[a,b]$ by setting

$$\rho(f,g) = \int_a^b |f(x) - g(x)| dx.$$

   Show that $\rho(f,g)$ is a metric. (It is called the L^1-metric.)

4. Let x_n be a sequence of points in $\mathbb{R}^m$, and let $x \in \mathbb{R}^m$. Show that $x_n \to x$ in Euclidian metric if and only if $x_n \to x$ in Manhattan metric. Recall that the Euclidian metric is given by

$$\rho(x,y) = \sqrt{\sum_{i=1}^{n} (x^i - y^i)^2}.$$

5. Explain why uniform convergence of a sequence of functions implies pointwise convergence. Recall that a sequence of functions $f_n \in C[a,b]$ is said to converge pointwise to a function $f \in C[a,b]$ if $f_n(x) \to f(x)$ for any $x \in X$, while uniform convergence is convergence in the metric

$$\rho(f,g) = \sup_{[a,b]} |f - g|.$$

6. Prove that uniform convergence implies convergence in L^1-metric.

7. Consider a sequence of continuous functions on $[-1,1]$ given by

$$f_n(x) = \begin{cases} 0, \text{ if } x \leq -\frac{1}{n}, \\ 1 + nx \text{ if } -\frac{1}{n} < x \leq 0, \\ 1 - nx \text{ if } 0 < x \leq \frac{1}{n}, \\ 0, \text{ if } x > \frac{1}{n}. \end{cases}$$

   Show that $f_n \to 0$ in L^1-metric, but not pointwise.

8. Consider a sequence of continuous functions on $[0,1]$ given by

$$f_n(x) = \begin{cases} 0, \text{ if } x \leq 1 - \frac{1}{n}, \\ n + 2n^2(x - (1 - \frac{1}{2n})) \text{ if } 1 - 1/n < x \leq 1 - \frac{1}{2n}, \\ n - 2n^2(x - (1 - \frac{1}{2n})) \text{ if } 1 - \frac{1}{2n} < x \leq 1. \end{cases}$$

Show that $f_n \to 0$ pointwise, but not in L^1-metric.

9. Prove that a closed interval $[a,b]$ is a complete metric space with respect to the standard distance function $|x-y|$.

10. Prove that $\mathbb{R}^n$ is a complete metric space with respect to the Euclidian metric.