MAT337H1, Introduction to Real Analysis: Solution to Problem 7 for Mar 22 class

Problem. Consider a sequence of continuous functions on $[-1,1]$ given by

$$f_n(x) = \begin{cases} 0, \text{ if } x \le -\frac{1}{n}, \\ 1+nx \text{ if } -\frac{1}{n} < x \le 0, \\ 1-nx \text{ if } 0 < x \le \frac{1}{n}, \\ 0, \text{ if } x > \frac{1}{n}. \end{cases}$$

Show that $f_n \to 0$ in L^1-metric, but not pointwise.

Solution. Let ρ be the L^1-metric on the set $C[-1,1]$ of continuous functions on $[-1,1]$. To prove that $f_n \to 0$ in L^1-metric, we need to show that $\rho(f_n, 0) \to 0$ when $n \to \infty$. We have

$$\rho(f_n, 0) = \int_{-1}^{1} |f_n(x) - 0| dx = \int_{-1}^{1} |f_n(x)| dx \overset{\text{since} f_n \ge 0}{=} \int_{-1}^{1} f_n(x) dx.$$

The latter integral is equal to the area under the graph of f_n, i.e., the area of the triangle with vertices $(-\frac{1}{n}, 0), (\frac{1}{n}, 0), (0,1)$. So,

$$\int_{-1}^{1} f_n(x) dx = \frac{1}{n}.$$

(This can also be computed using the fundamental theorem of calculus.) Hence we see that $\rho(f_n, 0) = \frac{1}{n} \to 0$, meaning that $f_n \to 0$ in L^1-metric.

As for pointwise convergence, notice that $f_n(0) = 1$ for any n, so

$$\lim_{n\to\infty} f_n(0) = 1 \ne 0,$$

meaning that the sequence f_n does not converge to 0 pointwise. (Recall that pointwise convergence of f_n to 0 in $[-1,1]$ means that $f_n(x)$ convergence to 0 for any $x \in [-1,1]$.)