MAT337H1, Introduction to Real Analysis: recommended problems for Mar 24 class

1. Consider a sequence of continuous functions on $[-1,1]$ given by

$$f_n(x) = \begin{cases} 0, \text{ if } x \le -\frac{1}{n}, \\ \frac{1}{2} + \frac{nx}{2}, \text{ if } -\frac{1}{n} < x \le \frac{1}{n}, \\ 1, \text{ if } x > \frac{1}{n}. \end{cases}$$

   Show that this sequence is Cauchy in the L^1-metric, but does not converge (in the same metric) to any continuous function. Therefore, the space $C[-1,1]$ with the L^1-metric is not complete.

2. Consider a sequence of continuous functions on $[0,1]$ given by $f_n(x) = x^n$. Let also f be a function on $[0,1]$ given by

$$f(x) = \begin{cases} 0, \text{ if } x < 1, \\ 1, \text{ if } x = 1. \end{cases}$$

   Show that $f_n \to f$ pointwise, but not uniformly.

3. Show that if (X, ρ) is metric space, and $x_n \in X$ is a convergent sequence (with respect to the metric ρ), then its limit is unique. In other words, for any sequence $x_n \in X$, there is at most one $x \in X$ such that $\lim_{n\to\infty} \rho(x_n, x) = 0$.

4. Show that for the functions f_n and f from Problem 2 we have $f_n \to f$ in the L^1-metric. Show that we also have $f_n \to 0$ in the L^1-metric. Explain why this does not contradict the result of Problem 3.

5. Show that the space of Riemann integrable functions on $[a,b]$ endowed with the uniform metric is complete. *Hint: use the result of Problem 6 for Mar 15 class.*

6. Show that the function $f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$ is well-defined and continuous on $(-1,1)$. *Hint: by definition of infinite sums, we have $f(x) = \lim_{n\to\infty} f_n(x)$, where $f_n(x) = \sum_{k=1}^{n} \frac{x^k}{k}$. Prove that for any $\lambda \in (0,1)$ the sequence f_n is Cauchy with respect to the uniform metric on $C[-\lambda, \lambda]$.*

   Remark: One can show that $f(x) = \ln(1-x)$ for $x \in [-1,1)$.