MAT337H1, Introduction to Real Analysis: solution to Problem 1 for Mar 24 class

Problem. Consider a sequence of continuous functions on $[-1, 1]$ given by

$$f_n(x) = \begin{cases} 0, \text{ if } x \leq -\frac{1}{n}, \\ \frac{1}{2} + \frac{nx}{2}, \text{ if } -\frac{1}{n} < x \leq \frac{1}{n}, \\ 1, \text{ if } x > \frac{1}{n}. \end{cases}$$

Show that this sequence is Cauchy in the L^1-metric, but does not converge (in the same metric) to any continuous function. Therefore, the space $C[-1, 1]$ with the L^1-metric is not complete.

Solution. Let m, n be natural numbers. Assume that $n \geq m$. Then $f_n - f_m = 0$ when $|x| \geq \frac{1}{m}$. So,

$$\int_{-1}^{1} |f_n(x) - f_m(x)| dx = \int_{-\frac{1}{m}}^{\frac{1}{m}} |f_n(x) - f_m(x)| dx.$$

Furthermore, we have

$$|f_n(x) - f_m(x)| \leq |f_n(x)| + |f_m(x)| \leq 2,$$

and it follows that

$$\int_{-\frac{1}{m}}^{\frac{1}{m}} |f_n(x) - f_m(x)| dx \leq \int_{-\frac{1}{m}}^{\frac{1}{m}} 2dx = \frac{4}{m},$$

where we used that $f \geq g$ implies $\int_a^b f dx \geq \int_a^b g dx$ for any $a < b$ and any functions f, g integrable on $[a, b]$ (Exercise H for Section 6.3). So, for $n \geq m$ we have $\rho(f_n, f_m) \leq \frac{4}{m}$, where ρ is the L^1-metric. Analogously, for $m \geq n$, we have $\rho(f_n, f_m) \leq \frac{4}{n}$. Therefore, if $m, n \geq N$, then $\rho(f_n, f_m) \leq \frac{4}{N}$. Since the latter expression can be made less than ε for any $\varepsilon > 0$, it follows that the sequence f_n is Cauchy.

To show that the sequence f_n does not converge we use the following lemma. The lemma should be familiar if you solved Problem 3 for Mar 22 class.

Lemma. Let $f(x)$ be a non-negative continuous function on a closed interval $[a, b]$. Assume that $\int_a^b f(x)dx = 0$. Then $f(x) = 0$ in $[a, b]$.

Proof of the lemma. Assume that there exists $x_0 \in [a, b]$ such that $f(x_0) > 0$. Then, by continuity of f, there is $\delta > 0$ such that $|f(x) - f(x_0)| < \frac{1}{2} f(x_0)$ for any $x \in [a, b]$ satisfying $|x - x_0| < \delta$. Let $[c, d]$ be any closed interval completely contained in the intersection $[a, b] \cap (x_0 - \delta, x_0 + \delta)$. (For instance, one can take $c = \max(a, x_0 - \frac{1}{2}\delta)$, $d = \min(b, x_0 + \frac{1}{2}\delta)$.) Then, for any $x \in [c, d]$ we have

$$|f(x) - f(x_0)| < \frac{1}{2} f(x_0) \quad \Rightarrow \quad f(x) > \frac{1}{2} f(x_0).$$

Therefore,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx + \int_d^b f(x)dx \geq \int_c^d f(x)dx$$
$$\geq \int_c^d \frac{1}{2} f(x_0)dx = \frac{1}{2} f(x_0)(c-d),$$

where in the first inequality we used that the integral of a non-negative function is non-negative, while in the second inequality we used that $f \geq g$ implies $\int_c^d f dx \geq \int_c^d g dx$ for any $c < d$ and any functions f, g integrable on $[c,d]$ (Exercise H for Section 6.3). The obtained inequality contradicts the assumption $\int_a^b f(x)dx = 0$, so the lemma is proved.

Now we show that the sequence f_n does not converge.

Proof 1. Assume that $f_n \to f$, where f is continuous. This means that

$$\lim_{n\to\infty} \int_{-1}^1 |f_n(x) - f(x)|dx = 0.$$

Note that for any subinterval $[a,b] \subset [-1,1]$ we have

$$\int_a^b |f_n(x) - f(x)|dx \leq \int_{-1}^1 |f_n(x) - f(x)|dx,$$

so

$$\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|dx = 0$$

by the squeeze theorem. Applying this for $a = -1$ and any $b \in (-1, 0)$, we get

$$\lim_{n\to\infty} \int_{-1}^b |f_n(x) - f(x)|dx = 0.$$

At the same time, there exists N such that $f_n = 0$ on $[-1, b]$ for $n \geq N$ (any $N \geq -\frac{1}{b}$ works). So, for $n \geq N$, we have

$$\int_{-1}^b |f_n(x) - f(x)|dx = \int_a^b |f(x)|dx.$$

Since the limit of the left-hand side as $n \to \infty$ is 0, we get that

$$\int_{-1}^b |f(x)|dx = 0,$$

and it follows from the lemma that $f = 0$ on $[-1, b]$. Since b is an arbitrary negative number, this means that $f = 0$ on $[-1, 0)$. Similarly, using

$$\lim_{n\to\infty} \int_a^1 |f_n(x) - f(x)|dx = 0,$$

where $a \in (0, 1)$, one shows that $f = 1$ on $(0, 1]$. But continuous functions equal to 0 on $[-1, 0)$ and equal to 1 on $(0, 1]$ do not exist. The obtained contradiction shows that the sequence f_n does not converge.

Proof 2. Let f be the function on $[-1, 1]$ equal to 0 for $x \leq 0$ and equal to 1 for $x > 0$. Consider the set X of functions on $[0, 1]$ that consists of all continuous functions and the function f: $X = C[-1, 1] \cup \{f\}$. Notice that the L^1-distance $\rho(g, h) = \int_{-1}^{1} |g - h| dx$ is well-defined on X. Indeed, for any $g, h \in X$ the function $|g - h|$ is bounded (why?) and continuous everywhere except, possibly, at 0, so it is integrable (Problem 7 for Mar 15). Moreover (X, ρ) is a metric space. The triangle inequality is checked in the same way as for continuous functions. So, what needs to be checked is that $\rho(g, h) \neq 0$ when $g \neq h$. We already know this is true when $g, h \in C[-1, 1]$, so it suffices to consider the case when g is continuous and $h = f$. Assume that $\rho(g, f) = 0$. But then

$$\int_{-1}^{0} |g - f| dx = 0.$$

Since $|g - f|$ is continuous in $[-1, 0]$, it follows that $g = f = 0$ on $[-1, 0]$. Analogously, $g = f = 1$ on $[a, 1]$ for any $a > 0$, which means that $g(x) = 1$ for $x > 0$. But continuous functions equal to 0 on $[-1, 0]$ and equal to 1 on $(0, 1]$ do not exist. So, it follows that $\rho(g, f) \neq 0$, and (X, ρ) is a metric space. In this metric space we have $f_n \to f$ (check this). But then f_n does not converge to a continuous function since the limit of a convergent sequence in a metric space is unique (Problem 3 for Mar 24).