MAT337H1, Introduction to Real Analysis: recommended problems for Mar 29 class

1. In class we used that
$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$
for a function $f \in C[a,b]$. Prove that this inequality in fact holds for any function f Riemann integrable on $[a,b]$. (In particular show that this inequality makes sense for any integrable f. In other words, if f is integrable, then $|f|$ is integrable as well.)

2. Prove that a mapping $F\colon X \to Y$ between metric spaces is continuous if and only if for any sequence x_n convergent to $x \in X$ the sequence $F(x_n)$ converges to $F(x)$.

3. Prove the following generalization of the triangle inequality ("polygon inequality"): If (X,ρ) is metric space, and $x_1,\dots,x_n \in X$, then
$$\rho(x_1,x_n) \leq \sum_{k=1}^{n-1} \rho(x_k,x_{k+1}).$$

4. Consider a sequence of continuous functions on $[0,1]$ given by
$$f_n(x) = \begin{cases} \sqrt{n}(1-xn), & \text{if } x \leq \frac{1}{n}, \\ 0, & \text{if } x > \frac{1}{n}. \end{cases}$$
Show that $f_n \to 0$ in the L^1-metric, but $f_n^2 \not\to 0$ in the L^1-metric. Hence show that the mapping $F : C[0,1] \to C[0,1]$ given by $F(f) = f^2$ is discontinuous at $0 \in C[0,1]$ when both the domain and the codomain are endowed with the L^1-metric.

5. Show that the mapping F from the previous problem is in fact discontinuous at every "point" $f \in C[0,1]$. Hint: Show that $f_n + f \to f$, but $(f_n+f)^2 \not\to f^2$.

6. Let (X,ρ) be a metric space, and let $x_0 \in X$. Show that the function $f\colon X \to \mathbb{R}$ given by $f(x) = \rho(x,x_0)$ (i.e., a function measuring the distance to a given point) is continuous with respect to the metric ρ.

7. Let $f(x)$ be a differentiable function on $\mathbb{R}$. Show that f is a contraction if and only if $f'(x)$ is bounded, and $\sup|f'| < 1$.

8. Let F be a contraction on a complete metric space (X,ρ): $\rho(F(x),F(y)) \leq k\rho(x,y)$, $k \in [0,1)$. Let x^* be the fixed point of F. Take arbitrary $x_0 \in X$, and define a sequence x_n by the rule $x_n = f(x_{n-1})$. Show that
$$\rho(x_n,x^*) \leq \frac{k^n}{1-k}\rho(x_0,x_1).$$

9. Use the contraction mapping principle to show that the equation $\cos(x) = x$ has exactly one real solution.

10. Let $\lambda \in (0,1)$. Consider the metric space $C[0,\lambda]$ with the uniform metric. Show that $F\colon C[0,\lambda] \to C[0,\lambda]$ given by

$$(F(f))(x) = 1 + \int_0^x f(t)dt$$

is a contraction. Find the fixed point of the map F. By applying the map n times to $f = 1$, find a sequence converging to the fixed point uniformly in $[0,\lambda]$.