MAT337H1, Introduction to Real Analysis: recommended problems for Mar 31 class

1. Let $f(x)$ be a function defined and continuous in an interval (a,b). Let also $x_0 \in (a,b)$, and let t_0 be any real number. Show that the initial value problem

$$\frac{dx}{dt} = f(x), \quad x(t_0) = x_0$$

has a solution $x(t)$ defined for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ for some $\varepsilon > 0$.

Hint: If $f(x_0) = 0$, then the desired solution is $x(t) \equiv x_0$. If $f(x_0) \neq 0$, then $f(x) \neq 0$ is some neigborhood of x_0. So, in that neigborhood we should have

$$\frac{x'(t)}{f(x(t))} = 1 \quad \Rightarrow \quad \int_{t_0}^{t} \frac{x'(\xi)}{f(x(\xi))} d\xi = t - t_0.$$

Further, since $x'(\xi) = f(x(\xi)) \neq 0$, we can apply the substitution rule and get

$$\int_{t_0}^{t} \frac{x'(\xi)}{f(x(\xi))} d\xi = \int_{x_0}^{x(t)} \frac{d\eta}{f(\eta)}.$$

This argument shows that if a solution $x(t)$ exists, then it satisifes

$$t - t_0 = \int_{x_0}^{x(t)} \frac{d\eta}{f(\eta)}.$$

Show that this equation indeed determines a function $x(t)$ defined for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ for some $\varepsilon > 0$, and that this function solves the given initial value problem.

Remark: In elementary calculus or differential equations courses, the above computation is usually performed in the following, not quite formal, way:

$$\frac{dx}{dt} = f(x) \quad \Rightarrow \quad \frac{dx}{f(x)} = dt \quad \Rightarrow \quad \int \frac{dx}{f(x)} = \int dt,$$

which gives the same result as above. This is known as the *separation of variables* method. More generally, it applies to equations of the form

$$\frac{dx}{dt} = f(x)g(t).$$

2. Apply the above method to solve the equation

$$x' = 3x^{\frac{2}{3}}.$$

Show that the solution of the initial value problem $x(0) = 0$ is not unique.

Remark: Recall that Picard-Lindelof theorem, in particular, requires, that the right-hand side of the equation $x' = f(x,t)$ is differentiable in x. This condition is violated in this case, that is why it is possible to have more than one solution.