MAT337, Real Analysis Midterm 1 Feb. 8, 2017 2:10 - 4pm

Total: 50 points. Each problem is worth 10 points. No aids allowed!

Note: It is not necessary to reprove facts which are proved in the textbook or have been proved in class. However, you should always state the fact that you are using. (If it is a theorem which has a name, it is sufficient to state the name.)

1. (a) Give a definition of a Cauchy sequence.
   (b) Let a_n be a Cauchy sequence such that $a_n \neq 0$ for every n. Is it always true that $1/a_n$ is also a Cauchy sequence? Justify your answer. (Prove if true, give a counterexample if not.)

2. (a) Give a definition of a bounded set.
   (b) Is the set
   $$\left\{ \frac{1}{x^2 - 3} \mid x \in \mathbb{Q} \right\}$$
   bounded? Justify your answer.

3. Define a sequence a_n by
   $$a_n = \frac{n^2 + 1}{n^2 - 5n + 7}.$$
   (a) Prove that for any real number $\varepsilon > 0$ there exists a natural number N such that $|a_n - 1| < \varepsilon$ for any natural number $n \geq N$. (You should explicitly find N in terms of ε.)
   (b) What does the result of (a) say about the limit of a_n?

4. Let $X = \{x \in \mathbb{R} \mid x^3 + x < 1\}$.
   (a) Show that the set X is nonempty and bounded above, and hence has a least upper bound.
   (b) Prove that $(\sup X)^3 + \sup X = 1$.

5. Define a sequence x_n by
   $$x_1 = 2, \quad x_{n+1} = \frac{1}{x_n} + \frac{x_n}{2} \ \forall\ n \geq 1.$$
   (a) Using the monotone convergence theorem, or otherwise, prove that the sequence x_n converges.
   (b) What is the limit of x_n? Justify your answer. (You should prove that the limit is a given number, however it is not necessary to do it by definition. You can use properties of limits.)