MAT337, Real Analysis   Midterm 1   Solutions

1. (a) Give a definition of a Cauchy sequence.

   A sequence a_n os real numbers is called Cauchy if for any real number $\varepsilon > 0$ there exists a natural number N such that $|a_n - a_m| < \varepsilon$ as long as $m, n \geq N$.

   (b) Let a_n be a Cauchy sequence such that $a_n \neq 0$ for every n. Is it always true that $1/a_n$ is also a Cauchy sequence? Justify your answer. (Prove if true, give a counterexample if not.)

   This is not always true. For example, the sequence $a_n = 1/n$ is Cauchy (because it converges to 0), but the sequence $1/a_n = n$ is not Cauchy (because is it is unbounded and hence divergent).

2. (a) Give a definition of a bounded set.

   A subset $S \subset \mathbb{R}$ is called bounded if there exist numbers $m, M \in \mathbb{R}$ such that $m \leq x \leq M$ for any $x \in S$. (Equivalently, if there exists $c > 0$ such that $|x| \leq c$ for any $x \in S$.)

   (b) Is the set

   $$\left\{ \frac{1}{x^2 - 3} \mid x \in \mathbb{Q} \right\}$$

   bounded? Justify your answer.

   No, this set is not bounded. Take a sequence x_n of rational numbers such that $\lim_{n\to\infty} x_n = \sqrt{3}$. Then $\lim_{n\to\infty} x_n^2 = 3$ and $\lim_{n\to\infty}(x_n^2 - 3) = 0$. This, in particular, means that for any $\varepsilon > 0$ there exists n such that $|x_n^2 - 3| < \varepsilon$, which implies $\left|\frac{1}{x_n^2-3}\right| > \frac{1}{\epsilon}$. Notice that $\frac{1}{x_n^2-3}$ is in our set. Furthermore, since $\varepsilon > 0$ is arbitrary, $\frac{1}{\epsilon}$ can take any positive value, so for any $c > 0$ there exists an element y of our set with $|y| > c$. This means that our set is unbounded.

3. Define a sequence a_n by

   $$a_n = \frac{n^2 + 1}{n^2 - 5n + 7}.$$

(a) Prove that for any real number $\varepsilon > 0$ there exists a natural number N such that $|a_n - 1| < \varepsilon$ for any natural number $n \geq N$. (You should explicitly find N in terms of ε.)

$$|a_n - 1| = \left| \frac{n^2+1}{n^2-5n+7} - 1 \right| = \left| \frac{5n-6}{n^2-5n+7} \right| = \frac{|5n-6|}{n^2-5n+7}$$

Notice that for $n \geq 2$ we have $|5n-6| = 5n-6 < 5n$. At the same time, for $n = 1$, we also have $|5n-6| < 5n$ $(1 < 5)$. So, we always have $|5n-6| < 5n$. Further, we have $n^2 - 5n + 7 > n^2 - 5n$. So, as long as $n^2 - 5n > 0$ (i.e., $n > 5$), we have

$$|a_n - 1| = \frac{|5n-6|}{n^2-5n+7} < \frac{5n}{n^2-5n} = \frac{5}{n-5}.$$

The expression $\frac{5}{n-5}$ is less than ϵ when $n > 5 + 5/\varepsilon$. So, as N we can take any integer greater than $5 + 5/\varepsilon$. (Notice that the latter expression is automatically greater than 5, so all our estimates are valid.)

(b) What does the result of (a) say about the limit of a_n?

This limit is equal to 1, by definition.

4. Let $X = \{x \in \mathbb{R} \mid x^3 + x < 1\}$.

(a) Show that the set X is nonempty and bounded above, and hence has a least upper bound.

We have $0^3 + 0 < 1$, so $0 \in X$. Therefore, X is not empty. Further, let us show that 1 is an upper bound for X. For this we need to prove that $x < 1$ for any $x \in X$. Assume that $x > 1$ for some $x \in X$. Then $x^3 > 1$, so $x + x^3 > 2$, which contradicts x being in X. So, 1 is indeed an upper bound.

(b) Prove that $(\sup X)^3 + \sup X = 1$.

This can be proved in a way similar to how we proved that if $z = \sup\{x \in \mathbb{R} \mid x^2 < 2, x > 0\}$, then $z^2 = 2$. (See lecture notes and recommended problems for Jan 18.) Let me give a computationally simpler proof using limits. Let $z = \sup X$. Consider the sequence $x_n = z + 1/n$. Note that $x_n \notin X$, since z is an upper bound for S. Therefore, $x_n^3 + x_n \geq 1$. Further, notice that $\lim_{n\to\infty} x_n = z$, so $\lim_{n\to\infty}(x_n^3 + x_n) = z^3 + z$. But since $x_n^3 + x_n \geq 1$, it follows that $z^3 + z \geq 1$. (See Exercise C for Section

2.4.) Further, consider the sequence $y_n = z - 1/n$. Then y_n is not an upper bound for X (because z is the least upper bound). This means that there is $x \in X$ such that $y_n < x$. From the latter it follows that $y_n^3 < x^3$ and $y_n^3 + y_n < x_3 + x < 1$. So, we conclude that $y_n^3 + y_n < 1$. At the same time, using the same argument as for the sequence x_n, we get $\lim_{n\to\infty}(y_n^3 + y_n) = z^3 + z$, so $z^3 + z \leq 1$. (We are again using the result of Exercise C for Section 2.4.) So, we showed that $z^3 + z \geq 1$ and $z^3 + z \leq 1$, meaning that $z^3 + z = 1$.

5. Define a sequence x_n by

$$x_1 = 2, \quad x_{n+1} = \frac{1}{x_n} + \frac{x_n}{2} \ \forall\, n \geq 1.$$

(a) Using the monotone convergence theorem, or otherwise, prove that the sequence x_n converges.

Let us show that this sequence is non-increasing, i.e. that $x_{n+1} \leq x_n$ for every n. This inequality can be rewritten as

$$\frac{1}{x_n} + \frac{x_n}{2} \leq x_n \Leftrightarrow \frac{1}{x_n} \leq \frac{x_n}{2} \Leftrightarrow x_n^2 \geq 2 \Leftrightarrow x_n \geq \sqrt{2},$$

where in the last two steps we used that all terms of the sequence x_n are positive. (Note that x_1 is positive and positivity of x_n implies positivity of x_{n+1}, so all terms are positive by induction.) We conclude that the sequence is indeed non-increasing as long as it is true that $x_n \geq \sqrt{2}$ for every n. Let us check this. This will also show that the sequence x_n is bounded below.

Notice that $x_1 > \sqrt{2}$. So, it suffices to show that $x_n \geq \sqrt{2}$ for $n \geq 2$. For such values of n, we have

$$x_n - \sqrt{2} = \frac{1}{x_{n-1}} + \frac{x_{n-1}}{2} - \sqrt{2} = \frac{2 + x_{n-1}^2 - 2\sqrt{2}x_{n-1}}{2x_{n-1}} = \frac{(x_{n-1} - \sqrt{2})^2}{2x_{n-1}} \geq 0,$$

as desired. So, the sequence x_n is non-increasing and bounded below, and hence convergent by the monotone convergence theorem.

(b) What is the limit of x_n? Justify your answer. (You should prove that the limit is a given number, however it is not necessary to do it by definition. You can use properties of limits.)

Let $L = \lim_{n\to\infty} x_n$ (this limit exists by part (a)). Consider the sequence x_{n+1} (i.e., the sequence $x_2, x_3, \ldots$). It has the same limit L. On the other hand, by definition of x_n, this sequence is equal to the sequence $1/x_n + x_n/2$. By properties of limits, the limit of the latter sequence is $1/L + L/2$. (Here we use that $L \neq 0$. This follows, e.g., from the inequality $x_n \geq 2$.) Since the sequences x_{n+1} and $1/x_n + x_n/2$ are the same, their limits are equal:

$$L = \frac{1}{L} + \frac{L}{2}.$$

From this equation we find that $L = \pm\sqrt{2}$. But one cannot have $L = -\sqrt{2}$, since $x_n \geq 0$. So, $L = \sqrt{2}$.

This construction is known as the Babylonian method for finding square roots. In general, the sequence

$$x_{n+1} = \frac{1}{2}\left(\frac{a}{x_n} + x_n\right)$$

(where $a > 0$) converges to $\sqrt{a}$ (for any choice of $x_1 > 0$).