MAT337, Real Analysis Midterm 2 Mar 8, 2017 2:10 - 4pm

Total: 50 points. Each problem is worth 10 points. No aids allowed!

Note: It is not necessary to reprove facts which are proved in the textbook or have been proved in class. However, you should always state the fact that you are using. (If it is a theorem which has a name, it is sufficient to state the name.)

1. (a) Let f be a function on (a, b), and let $x_0 \in (a, b)$. Let also $L \in \mathbb{R}$. Define what it means that $\lim_{x\to x_0} f(x) = L$.

   (b) Using your definition, prove that there exists no finite limit $\lim_{x\to 0} \frac{1}{x}$.

2. (a) Let $f(x) = \max(1 - x/2, x)$. Show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x \in (\frac{2}{3} - \delta, \frac{2}{3} + \delta)$ we have $|f(x) - \frac{2}{3}| < \varepsilon$.

   (b) Hence determine whether $f(x)$ is continuous at $\frac{2}{3}$.

3. Let f be a function defined and continuous on $[0, 1]$. Assume also that $f(0) < 0$ and $f(1) > 0$. Let $x_0 = \inf\{x \in [0, 1] \mid f(x) > 0\}$. Prove that $f(x_0) = 0$.

4. Let $f(x)$ be a function defined for any $x \in \mathbb{R}$ and continuous on $(0, 1)$. Assume also that there exist (finite) limits $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$. Prove that $f(x)$ is bounded on $(0, 1)$.

5. Let f be a differentiable function on $\mathbb{R}$ which has infinitely many zeros (i.e., there are infinitely many points $x \in \mathbb{R}$ such that $f(x) = 0$). Show that its derivative $f'(x)$ also has infinitely many zeroes.

   Hint: One way to approach this problem is the following. First show that f' has at least one zero. Then assume that f' has finitely many zeros and derive a contradiction.