MAT337, Real Analysis   Midterm 2 – Solutions

1. (a) Let f be a function on (a,b), and let $x_0 \in (a,b)$. Let also $L \in \mathbb{R}$. Define what it means that $\lim_{x\to x_0} f(x) = L$.

   Solution. We say that $\lim_{x\to x_0} f(x) = L$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x \in (a,b)$ satisfying $0 < |x - x_0| < \delta$ we have $|f(x) - L| < \varepsilon$. (The condition $x \in (a,b)$ can be omitted, because when we say that $|f(x) - L| < \varepsilon$, this, in particular, means that $f(x)$ is well-defined.)

   (b) Using your definition, prove that there exists no finite limit $\lim_{x\to 0} \frac{1}{x}$.

   Solution. Assume that $\lim_{x\to 0} \frac{1}{x} = L$. Applying the above definition for $\varepsilon = 1$ (any other value of ε would work too) we see that there exists $\delta > 0$ such that for any x satisfying $0 < |x| < \delta$ we have $|\frac{1}{x} - L| < 1$, i.e., $\frac{1}{x} \in (L-1, L+1)$. But $0 < |x| < \delta$ is equivalent to $\frac{1}{x} \in (-\infty, -\frac{1}{\delta}) \cup (\frac{1}{\delta}, +\infty)$. So we get that any x in $(-\infty, -\frac{1}{\delta}) \cup (\frac{1}{\delta}, +\infty)$ is also in $(L-1, L+1)$. But this is not possible. For instance, $\max(L+2, \frac{1}{\delta}+1)$ is in the first set, but not in the second one. So, the definition of a limit cannot be satisfied, and there is no finite limit $\lim_{x\to 0} \frac{1}{x}$.

2. (a) Let $f(x) = \max(1 - x/2, x)$. Show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x \in (\frac{2}{3} - \delta, \frac{2}{3} + \delta)$ we have $|f(x) - \frac{2}{3}| < \varepsilon$.

   Solution. By definition of $f(x)$ we have $|f(x) - \frac{2}{3}| = |x - \frac{2}{3}|$ or $|f(x) - \frac{2}{3}| = |1 - \frac{x}{2} - \frac{2}{3}| = |\frac{1}{3} - \frac{x}{2}| = \frac{1}{2}|\frac{2}{3} - x| = \frac{1}{2}|x - \frac{2}{3}|$. So, if $|x - \frac{2}{3}| < \delta$, then $|f(x) - \frac{2}{3}| < \delta$. Therefore, $\delta = \varepsilon$ is what we are looking for.

   (b) Hence determine whether $f(x)$ is continuous at $\frac{2}{3}$.

   Solution. Since $f(\frac{2}{3}) = \frac{2}{3}$, the result of (a) means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any x satisfying $|x - \frac{2}{3}| < \delta$ we have $|f(x) - f(\frac{2}{3})| < \varepsilon$. But this means (by definition of continuity) that f is continuous at $\frac{2}{3}$.

3. Let f be a function defined and continuous on $[0,1]$. Assume also that $f(0) < 0$ and $f(1) > 0$. Let $x_0 = \inf\{x \in [0,1] \mid f(x) > 0\}$. Prove that $f(x_0) = 0$.

   Solution. Assume that $f(x_0) \neq 0$. Then, by definition of continuity at x_0 applied for $\varepsilon = |f(x_0)|$, there exists $\delta > 0$ such that for any $x \in [0,1]$ satisfying $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < |f(x_0)|$. The latter in particular means that $f(x)$ has the same sign as $f(x_0)$ when $x \in [0,1]$ and $|x - x_0| < \delta$ (cf. Exercise 4 for Feb 4 class).

Now consider two cases: $f(x_0) > 0$ and $f(x_0) < 0$. If $f(x_0) > 0$, we get that $f(x) > 0$ when $x \in [0,1]$ and $|x - x_0| < \delta$. Notice that since $f(0) < 0$, we have that 0 is not in the δ-neighborhood of x_0, and it follows that $x_0 - \frac{\delta}{2} > 0$. Since $x_0 - \frac{\delta}{2}$ is both in $[0,1]$ and δ-neighborhood of x_0, we get that $f(x_0 - \frac{\delta}{2}) > 0$, which contradicts x_0 being the infinum of $\{x \in [0,1] \mid f(x) > 0\}$.

Similarly, if $f(x_0) < 0$, we have that $f(x) < 0$ when $x \in [0,1]$ and $|x - x_0| < \delta$. Since $f(1) > 0$, it follows that $x_0 + \delta \leq 1$, and the interval $[x_0, x_0 + \delta)$ is completely contained in $[0,1]$. Therefore, f is well-defined and negative on $[x_0, x_0 + \delta)$. But by definition of x_0, the function f is non-positive on $[0, x_0)$ as well. So, we have that $f \leq 0$ on $[0, x_0 + \delta)$, and $x_0 + \delta$ is a lower bound for $\{x \in [0,1] \mid f(x) > 0\}$. But this contradicts x_0 being the greatest lower bound of this set.

So, we get a contradiction in both cases $f(x_0) > 0$ and $f(x_0) < 0$, and it follows that $f(x_0) = 0$.

4. Let $f(x)$ be a function defined for any $x \in \mathbb{R}$ and continuous on $(0,1)$. Assume also that there exist (finite) limits $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$. Prove that $f(x)$ is bounded on $(0,1)$.

   Solution. Let $L_0 = \lim_{x\to 0} f(x)$. Applying the definition of limit for $\varepsilon = 1$, we get that there exists δ such that for any $x \in (0,\delta)$ we have $|f(x) - L_0| < 1$, i.e., $L_0 - 1 < f(x) < L_0 + 1$.

   Similarly, let $L_1 = \lim_{x\to 1} f(x)$. Then there exists δ' such that for any $x \in (1-\delta', 1)$ we have $L_1 - 1 < f(x) < L_1 + 1$.

   So, $f(x)$ is bounded in $(0,\delta)$ and $(1-\delta', 1)$. Furthermore, $f(x)$ is continuous in $[\delta, 1-\delta']$, so it is bounded in that interval as well. (By making δ and δ' smaller if necessary, we can make δ smaller than $1-\delta'$, so that the interval $[\delta, 1-\delta']$ is well-defined.) Then it follows that f is bounded in $(0,1)$. Namely, if m and M are some lower and upper bounds for f in $[\delta, 1-\delta']$, then

   $$\min(m, L_0 - 1, L_1 - 1) \leq f(x) \leq \max(M, L_0 + 1, L_1 + 1)$$

   for any $x \in (0,1)$.

5. Let f be a differentiable function on $\mathbb{R}$ which has infinitely many zeros (i.e., there are infinitely many points $x \in \mathbb{R}$ such that $f(x) = 0$). Show that its derivative $f'(x)$ also has infinitely many zeroes.

   Solution. Assume that f' has finitely many, say, k, zeros (where k is a non-negative integer). Consider the $k+1$ intervals $I_1, \ldots, I_{k+1}$ obtained by removing the zeros of f' from the real line. Then f has at most one zero in each of the intervals I_j. Indeed, if there are two zeros of f in I_j, then by Rolle's theorem there is a zero of f' between them, which contradicts the construction of I_j's.

So, f has at most one zero in each of the intervals I_j, and, in addition, it may vanish at the zeros of f'. It follows that f has at most $2k + 1$ zeros in total. (One can improve this estimate to make it $k + 1$, but this is not needed.) This contradicts f having infinitely many zeros. The obtained contradiction shows that f' has infinitely many zeros.