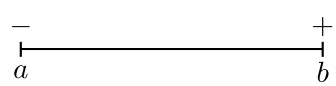
MAT 337 – PROBLEM SET
Due Wednesday, March 1 at 2:10pm

RULES:

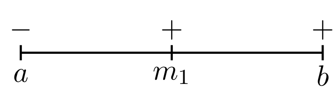
- There will be a 10% penalty for problem sets submitted on March 1 after 2:10 and before 5 pm. No problem sets will be accepted after March 1, 5pm.
- Students are expected to write up solutions independently. If several students hand in solutions which are so similar that one or more must have copied from someone else's solution, this will be treated as academic misconduct and reported to the administration.
- In order to receive full marks for computational questions, all details of the computation should be included in the solution. Even if the correct final answer is given, marks will be deducted if some details are left out.
- In solving questions involving proofs, it is not necessary to reprove facts that have been proved in class or in the sections of the text that have been covered.

The aim of this problem set is to investigate an alternative, more constructive, approach to the intermediate value theorem. Let f be a function continuous on an interval $[a, b]$. Then the intermediate value theorem says that for any ξ strictly between $f(a)$ and $f(b)$, there exists $c \in [a, b]$ such that $f(c) = \xi$. We already know that it is sufficient to prove this in the case $f(a) < f(b)$. (If $f(a) > f(b)$, one considers the function $g = -f$, which satisfies $g(a) < g(b)$.)

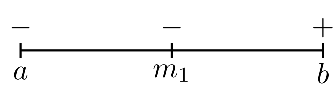
So, let $f(a) < f(b)$. This implies $f(a) < \xi < f(b)$. For any point $x \in [a, b]$, we shall write a "+" near this point if $f(x) > \xi$, and "-" if $f(x) < \xi$. Doing this for points a and b, we get the following picture:



Let $I_1 = [a, b]$. This interval has a property that the signs written at its endpoints are opposite. Now, we will construct a sequence of nested intervals with this property. Consider the midpoint m_1 of this interval I_1. Then, either $f(m_1) = \xi$, in which case we have proved the intermediate value theorem, or we can assign a sign to m_1 according to the above rule ("+" if $f(m_1) > \xi$ and "-" if $f(m_1) < \xi$). Then we get one of the following pictures:



or



In both cases, there is an interval with opposite signs at the endpoints: it is $[a, m_1]$ for the first picture and $[m_1, b]$ for the second. We shall call this interval I_2. Note that $I_2 \subset I_1$ by construction. Further, we repeat the same procedure for I_2: its midpoint m_2 either satisfies $f(m_2) = \xi$, or we can find an interval $I_3 \subset I_2$ (having m_2 as one of its endpoints) with opposite signs assigned to its endpoints.

Proceeding in the same fashion, we get a sequence of nested intervals $I_1 \supset I_2 \supset I_3 \supset \dots$. If this process terminates, this means that we found c such that $f(c) = \xi$. Otherwise, the sequence of nested intervals will be infinite.

1. Assume that the described process does not terminate after finitely many steps.
   (a) Prove that the intersection $\bigcap_{n\geq 1} I_n$ consists of one element.
   (b) Let c be the only element of $\bigcap_{n\geq 1} I_n$. Show that there exists a sequence $x_n \to c$ such that $f(x_n) < \xi$ for any n.
   (c) Show that there exists a sequence $\tilde{x}_n \to c$ such that $f(\tilde{x}_n) > \xi$ for any n.
   (d) Hence prove that $f(c) = \xi$.

2. (a) Assume that we apply the above procedure to $f(x) = x^2$, $[a, b] = [1, 2]$, and $\xi = 2$. Will the process terminate after finitely many steps?
   (b) By applying the above procedure to $f(x) = x^2$, $[a, b] = [1, 2]$, and $\xi = 2$, compute $\sqrt{2}$ with precision 10^{-3}. You should prove that the number $\tilde{c}$ you found indeed satisfies $|\tilde{c} - \sqrt{2}| < 10^{-3}$. (You cannot use the "actual" value of $\sqrt{2}$ generated by a computer / calculator.)
   (c) Compare this algorithm for computing $\sqrt{2}$ with the algorithm given by Problem 5 in the term test. Which algorithm allows you to compute $\sqrt{2}$ with precision 10^{-100} in a smaller number of steps?