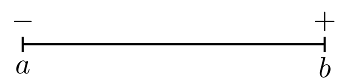
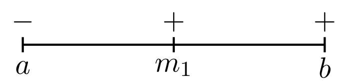
# MAT 337 – PROBLEM SET SOLUTIONS

The aim of this problem set is to investigate an alternative, more constructive, approach to the intermediate value theorem. Let $f$ be a function continuous on an interval $[a, b]$. Then the intermediate value theorem says that for any $\xi$ strictly between $f(a)$ and $f(b)$, there exists $c \in [a, b]$ such that $f(c) = \xi$. We already know that it is sufficient to prove this in the case $f(a) < f(b)$. (If $f(a) > f(b)$, one considers the function $g = -f$, which satisfies $g(a) < g(b)$.)

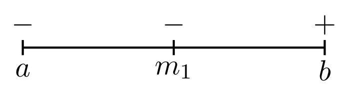
So, let $f(a) < f(b)$. This implies $f(a) < \xi < f(b)$. For any point $x \in [a, b]$, we shall write a "+" near this point if $f(x) > \xi$, and "-" if $f(x) < \xi$. Doing this for points $a$ and $b$, we get the following picture:




Let $I_1 = [a, b]$. This interval has a property that the signs written at its endpoints are opposite. Now, we will construct a sequence of nested intervals with this property. Consider the midpoint $m_1$ of this interval $I_1$. Then, either $f(m_1) = \xi$, in which case we have proved the intermediate value theorem, or we can assign a sign to $m_1$ according to the above rule ("+" if $f(m_1) > \xi$ and "-" if $f(m_1) < \xi$). Then we get one of the following pictures:




or




In both cases, there is an interval with opposite signs at the endpoints: it is $[a, m_1]$ for the first picture and $[m_1, b]$ for the second. We shall call this interval $I_2$. Note that $I_2 \subset I_1$ by construction. Further, we repeat the same procedure for $I_2$: its midpoint $m_2$ either satisfies $f(m_2) = \xi$, or we can find an interval $I_3 \subset I_2$ (having $m_2$ as one of its endpoints) with opposite signs assigned to its endpoints.

Proceeding in the same fashion, we get a sequence of nested intervals $I_1 \supset I_2 \supset I_3 \supset \dots$. If this process terminates, this means that we found $c$ such that $f(c) = \xi$. Otherwise, the sequence of nested intervals will be infinite.

1. Assume that the described process does not terminate after finitely many steps.

   (a) Prove that the intersection $\bigcap_{n \geq 1} I_n$ consists of one element.

   **Solution**. Let $l_n$ be the length of the interval $I_n$. Then $l_1 = b - a$, $l_2 = \frac{1}{2}(b - a)$, and, more generally, $l_{n+1} = \frac{1}{2} l_n$. It follows by induction that $l_n = \frac{1}{2^{n-1}}(b - a)$. In particular, $\lim_{n \to \infty} l_n = 0$. (Prove this.) So, by Problem 2(b) for Jan 27 class, we have that the intersection $\bigcap_{n \geq 1} I_n$ consists of one element.

(b) Let $c$ be the only element of $\bigcap_{n\geq 1} I_n$. Show that there exists a sequence $x_n \to c$ such that $f(x_n) < \xi$ for any $n$.

**Solution**. By construction, we assign different signs to endpoints of each interval $I_n$. This means that $f < \xi$ at one of the endpoints, and $f > \xi$ at the other one. Let $x_n$ be the endpoint of $I_n$ with $f < \xi$. Then $f(x_n) < \xi$. Furthermore, both $x_n$ and $c$ lie in the interval $I_n$, so $|x_n - c| \leq l_n$, and $-l_n \leq x_n - c \leq l_n$. Since $\lim_{n\to\infty} l_n = 0$, it follows by the squeeze theorem that $\lim_{n\to\infty}(x_n - c) = 0$, and thus $\lim_{n\to\infty} x_n = c$.

(c) Show that there exists a sequence $\tilde{x}_n \to c$ such that $f(\tilde{x}_n) > \xi$ for any $n$.

**Solution**. The construction is the same as in (b), but we take $\tilde{x}_n$ to be the endpoint of $I_n$ at which $f > \xi$.

(d) Hence prove that $f(c) = \xi$.

**Solution**. For the sequence $x_n$ constructed in (b), we have $\lim_{n\to\infty} x_n = c$. Since $f$ is continuous, it follows that $\lim_{n\to\infty} f(x_n) = f(c)$. At the same time, we have $f(x_n) < \xi$, so it follows that $\lim_{n\to\infty} f(x_n) \leq \xi$ (see Exercise C for Section 2.4), i.e., $f(c) \leq \xi$. Applying the same argument to the sequence $\tilde{x}_n$ constructed in part (c), we also get that $f(c) \geq \xi$. So $f(c) = \xi$.

2. (a) Assume that we apply the above procedure to $f(x) = x^2$, $[a, b] = [1, 2]$, and $\xi = 2$. Will the process terminate after finitely many steps?

**Solution**. No. The process terminates on $n$'th step if the midpoint $x$ of the interval $I_n$ satisfies $f(x) = \xi$. In our case this means $x^2 = 2$. But since we start with the interval $[1, 2]$ and successively take midpoints, the midpoint of $I_n$ is a rational number. At the same time, we know that there are no rational solutions to the equation $x^2 = 2$. So, the process will not terminate after finitely many steps.

(b) By applying the above procedure to $f(x) = x^2$, $[a, b] = [1, 2]$, and $\xi = 2$, compute $\sqrt{2}$ with precision $10^{-3}$. You should prove that the number $\tilde{c}$ you found indeed satisfies $|\tilde{c} - \sqrt{2}| < 10^{-3}$. (You cannot use the "actual" value of $\sqrt{2}$ generated by a computer / calculator.)

**Solution**. By construction, we have that $\sqrt{2} \in I_n$ for every $n$. Therefore, if we take $\tilde{c}$ to be the midpoint of $I_n$, then

$$|\tilde{c} - \sqrt{2}| \leq \frac{1}{2} l_n = \frac{1}{2^n}.$$

So, if we take $n = 10$, we have

$$|\tilde{c} - \sqrt{2}| \leq \frac{1}{1024} < 10^{-3},$$

as desired. We conclude that as $\tilde{c}$ we can take the midpoint of the interval $I_{10}$. We have $I_1 = [1, 2]$. Since for the midpoint $\frac{3}{2}$ we have $(\frac{3}{2})^2 = \frac{9}{4} > 2$, we should take $I_2 = [1, \frac{3}{2}]$.

Further,

$$\begin{aligned}
(\tfrac{5}{4})^2 &= \tfrac{25}{16} < 2 \Rightarrow I_3 = [\tfrac{5}{4}, \tfrac{3}{2}],\\
(\tfrac{11}{8})^2 &= \tfrac{121}{64} < 2 \Rightarrow I_4 = [\tfrac{11}{8}, \tfrac{3}{2}],\\
(\tfrac{23}{16})^2 &= \tfrac{529}{256} > 2 \Rightarrow I_5 = [\tfrac{11}{8}, \tfrac{23}{16}],\\
(\tfrac{45}{32})^2 &= \tfrac{2025}{1024} < 2 \Rightarrow I_6 = [\tfrac{45}{32}, \tfrac{23}{16}],\\
(\tfrac{91}{64})^2 &= \tfrac{8281}{4096} > 2 \Rightarrow I_7 = [\tfrac{45}{32}, \tfrac{91}{64}].\\
(\tfrac{181}{128})^2 &= \tfrac{32761}{16384} < 2 \Rightarrow I_8 = [\tfrac{181}{128}, \tfrac{91}{64}],\\
(\tfrac{363}{256})^2 &= \tfrac{131769}{65536} > 2 \Rightarrow I_9 = [\tfrac{181}{128}, \tfrac{363}{256}],\\
(\tfrac{725}{512})^2 &= \tfrac{565625}{262144} > 2 \Rightarrow I_{10} = [\tfrac{181}{128}, \tfrac{725}{512}]
\end{aligned}$$

So, $\sqrt{2} \approx \frac{1449}{1024}$ with precision $10^{-3}$.

(c) Compare this algorithm for computing $\sqrt{2}$ with the algorithm given by Problem 5 in the term test. Which algorithm allows you to compute $\sqrt{2}$ with precision $10^{-100}$ in a smaller number of steps?

**Solution**. As we already explained, the above algorithm gives precision $2^{-n}$ after $n$ steps. So, we will need $\log_2(10^{100}) \approx 333$ steps to get guaranteed precision $10^{-100}$. Now, let us estimate the precision of the algorithm from Problem 5 of the term test. We have

$$x_{n+1} - \sqrt{2} = \frac{1}{x_n} + \frac{x_n}{2} - \sqrt{2} = \frac{1}{2x_n}(2 - 2\sqrt{2}x_n + x_n^2) = \frac{1}{2x_n}(x_n - \sqrt{2})^2 < (x_n - \sqrt{2})^2,$$

where we used that $x_n > \sqrt{2} > \frac{1}{2}$. So, if $y_n = x_n - \sqrt{2}$ is the error on $n$'th step (we do not take the absolute value, since $x_n > \sqrt{2}$), then $y_{n+1} < y_n^2$, i.e., the precision is squared on each step. This is much better than what we had in the first algorithm. By induction, we get

$$y_n < y_1^{(2^{n-1})} = (2 - \sqrt{2})^{(2^{n-1})} < 0.6^{(2^{n-1})}$$

(the estimate $2 - \sqrt{2} < 0.6$ follows from the computation in (b)). So, we need

$$0.6^{(2^{n-1})} < 10^{-100} \Leftrightarrow 2^{n-1} > -100 \log_{0.6} 10 \approx 450,$$

so $n = 10$ steps is enough to get the desired precision. (Of course, our estimates are rather rough and can be improved.)