MATH534A, Problem Set 3, due Sep 18

All problems are worth the same number of points.

1. Prove that $\mathbb{R}P^1$ is diffeomorphic to S^1.

2. Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$ be the unit sphere, and let $p \in S^2$. Let also D be a 1'st order differential operator on S^2 at p, i.e. a rule that assigns a real number to any function which is defined and smooth in some neighborhood of p and has the following properties:

   - $D(cf) = cD(f)$ for any smooth function f and any $c \in \mathbb{R}$.
   - $D(f + g) = D(f) + D(g)$ in the intersection of domains of f and g.
   - $D(fg) = f(p)D(g) + D(f)g(p)$.

   Show that there exists a vector $v \in \mathbb{R}^3$ tangent to S^2 at p such that D is the differentiation in direction v.

3. The above construction defines an isomorphism i between T_pS^2 (defined as the space of 1'st order differential operator on S^2 at p) and the geometric tangent plane at p. Assume that p belongs to the northern hemisphere. Then we have a chart (x, y) around p, and the associated basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \in T_pM$. Find $i(\frac{\partial}{\partial x}), i(\frac{\partial}{\partial y})$.

4. Problem 3-8 from the textbook.