

### MATH534A, Problem Set 5, due Oct 9

All problems are worth the same number of points.

1. Prove that the set of points in  $\mathbb{R}^2$  satisfying the equation  $xy = 0$  is not a smooth submanifold of  $\mathbb{R}^2$ .
2. Let  $Z$  be a smooth manifold,  $Y$  be a smooth submanifold of  $Z$ , and  $X$  be a smooth submanifold of  $Y$ . Show that  $X$  is a smooth submanifold of  $Z$ . In other words, being a submanifold is a transitive relation.
3. (a) Show that the manifold  $\text{SO}_n(\mathbb{R})$ , consisting of  $n \times n$  real orthogonal matrices with determinant 1, is connected.  
(b) Show that the manifold  $\text{O}_n(\mathbb{R}) \setminus \text{SO}_n(\mathbb{R})$ , consisting of  $n \times n$  real orthogonal matrices with determinant  $-1$ , is diffeomorphic to  $\text{SO}_n(\mathbb{R})$  and hence also connected.
4. Show that  $\text{U}_n$ , the set of  $n \times n$  unitary matrices, is a smooth submanifold of  $\text{Mat}_{n \times n}(\mathbb{C}) \simeq \mathbb{R}^{2n^2}$  and find its dimension.
5. Prove that  $\text{SO}_2(\mathbb{R})$  is diffeomorphic to  $S^1$ .