1. Prove that the set of points in $\mathbb{R}^2$ satisfying the equation $xy = 0$ is not a smooth submanifold of $\mathbb{R}^2$.

   Solution: If this set of points were a smooth submanifold, then there would exist a function $f(x, y)$ smooth around $(0, 0)$ such that the gradient of f is not zero and such that $f(x, y) = 0$ on the set of points where $xy = 0$. Since f vanishes on both axes, we should have that both partial derivatives vanish at $(0, 0)$, but then the gradient of f would be zero there, contradicting our assumption.

   Another approach: This set of points is not even a topological manifold. We can see this by removing $(0, 0)$ from a connected open neighborhood U of the origin. The result is a set with four connected components, while if we remove a point from an open set in $\mathbb{R}$ or $\mathbb{R}^n$ for any n, we would be left with a set with 2 connected components (if we started with $\mathbb{R}$) or 1 component (for $n \geq 2$). Thus our neighborhood U is not homeomorphic to an open set in $\mathbb{R}^n$ for any n.

2. Let Z be a smooth manifold, Y a smooth submanifold of Z, and let X be a smooth submanifold of Y. Show that X is a smooth submanifold of Z. In other words, being a submanifold is a transitive relation.

   Solution: Let p be a point that lies in all three manifolds X, Y, and Z. Then there are local coordinates $z_1, \ldots, z_n$ on Z such that Y is given by $z_{m+1} = \cdots = z_n = 0$ near p. Furthermore, there are local coordinates $y_1, \ldots, y_m$ on Y such that X is given by $y_{k+1} = \cdots = y_m = 0$ near p.

   Now the coordinates $y_1, \ldots, y_m$ are functions of $z_1, \ldots, z_m$, so we can extend them to functions on an open set in Z since $z_1, \ldots, z_m$ are functions on an open set in Z. Then we take $y_1, \ldots, y_m, z_m + 1, \ldots, z_n$ as local coordinates for Z.

   We have that the Jacobian of the transformation $(z_1, \ldots, z_n) \mapsto (y_1, \ldots, y_m, z_{m+1}, \ldots, z_n)$ is equal to the Jacobian of the transformation $(z_1, \ldots, z_m) \mapsto (y_1, \ldots, y_m)$ and hence nonzero. Now in these new coordinates (namely $y_1, \ldots, y_m, z_{m+1}, \ldots z_n$), X is given inside Z (near p) by linear equations (namely the equations $y_{k+1} = \cdots = y_m = 0$) and hence X is a submanifold of Z.

3. (a) Show that the manifold $\mathrm{SO}_n(\mathbb{R})$, consisting of $n \times n$ real orthogonal matrices with determinant 1, is connected.

   Solution: By a theorem from linear algebra, we can write any orthogonal matrix A with determinant 1 as
   $$A = C^{-1} A_{can} C$$
   where A_{can} is block-diagonal with blocks
   $$\begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}$$
   (where the angles a may be different for different blocks) and possibly one 1×1 block with value 1. Now consider the matrix $C^{-1} A^t_{can} C$, where A^t_{can} is the same as A_{can} except in each block we have multiplied the angle a by t. Then for any $t \in [0, 1]$ this matrix lies in $\mathrm{SO}_n(\mathbb{R})$ and further for $t = 1$ this is our original matrix A while for $t = 0$, this matrix is the identity. Hence we have found a path in $\mathrm{SO}_n(\mathbb{R})$ connecting any matrix to the identity and thus $\mathrm{SO}_n(\mathbb{R})$ is path-connected and hence connected.

   (b) Show that the manifold $\mathrm{O}_n(\mathbb{R}) \setminus \mathrm{SO}_n(\mathbb{R})$, consisting of $n \times n$ real orthogonal matrices with determinant -1, is diffeomorphic to $\mathrm{SO}_n(\mathbb{R})$ and hence also connected.

   Solution: Let B be a fixed orthogonal matrix with determinant -1. Then the required diffeomorphism is $A \mapsto BA$. This map is smooth because it is the restriction of a linear map from the space of all $n \times n$ matrices to itself, where this linear map is given by the same formula. (See Corollary 2.5 of the online lecture notes.)

It remains to show that the inverse map $A \mapsto B^{-1}A$ is smooth as well. But this is true by exactly the same argument: that is, this map is again a restriction of a linear map $\mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathrm{Mat}_{n\times n}(\mathbb{R})$ and hence smooth.

4. Show that U_n, the set of $n \times n$ unitary matrices, is a smooth submanifold of $\mathrm{Mat}_{n\times n}(\mathbb{C}) \simeq \mathbb{R}^{2n^2}$ and find its dimension.

Solution: The mapping Φ given by $\Phi(A) = AA^*$ (where A^* is the conjugate transpose of A) takes all complex $n \times n$ matrices to Hermitian matrices (i.e. matrices satisfying $B^* = B$). Let $\gamma(t) = A + tX$ be a curve in $\mathrm{Mat}_{n\times n}(\mathbb{C})$ with $\gamma(0) = A$ and $\gamma'(0) = X$. We can calculate the differential of Φ as follows:

$$\begin{aligned} d_A\Phi(X) &= \frac{d}{dt}\bigg|_{t=0} (A + tX)(A + tX)^* \\ &= \frac{d}{dt}\bigg|_{t=0} (AA^* + tAX^* + tXA^* + t^2XX^*) \\ &= AX^* + XA^*. \end{aligned}$$

If we assume that A is non-degenerate, then this differential is a composition of two surjective maps, $X \mapsto XA^*$ and $X \mapsto X + X^*$ and hence is surjective. So our map Φ is a submersion (when restricted to non-degenerate matrices) and the preimage of the identity matrix must be a submanifold. Thus U_n is a submanifold of $\mathrm{Mat}_{n\times n}(\mathbb{C})$, as required.

Next we must find the dimension of U_n. We have just shown that $\Phi : \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathrm{Herm}_{n\times n}$ is a submersion with U_n as a level set ($\mathrm{Herm}_{n\times n}$ is the space of Hermitian matrices). Hence we have that $\dim \mathrm{U}_n = \dim \mathrm{Mat}_{n\times n}(\mathbb{C}) - \dim \mathrm{Herm}_{n\times n}$. Clearly $\dim \mathrm{Mat}_{n\times n}(\mathbb{C}) = 2n^2$, so we just need to find the dimension of the space of Hermitian matrices.

To see that $\mathrm{Herm}_{n\times n}$ has dimension n^2, note that we can write any Hermitian matrix as the sum of a (real) symmetric matrix and a (real) antisymmetric matrix multiplied by i. Now the space of symmetric matrices has dimension $n(n+1)/2$ while the space of antisymmetric matrices has dimension $n(n-1)/2$. Adding these two dimensions gives us n^2, as claimed.

Hence the dimension of U_n is $2n^2 - n^2 = n^2$.

5. Prove that $\mathrm{SO}_2(\mathbb{R})$ is diffeomorphic to S^1.

Solution: As a set, SO(2) is just

$$\left\{ \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix} : a \in \mathbb{R} \right\}.$$

Now given the circle S^1 in $\mathbb{R}^2$, we can map it to a set of matrices as follows: for any point $(x, y) \in S^1$,

$$(x, y) \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

This map is smooth because it is the restriction of a linear map $\mathbb{R}^2 \to \mathrm{Mat}_{2\times 2}(\mathbb{R})$ (see Corollary 2.5 in the online lecture notes) and this map is obviously a bijection between S^1 and $\mathrm{SO}_2(\mathbb{R})$ written as matrices. We can also write the inverse to this map:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, c)$$

This inverse map is also the restriction of a linear map, so it too is smooth and hence S^1 and $\mathrm{SO}_2(\mathbb{R})$ are diffeomorphic.