MATH534A, Problem Set 6, due October 16

All problems are worth the same number of points.

1. (a) Let V be an m-dimensional subspace of $\mathbb{R}^n$. Show that there exists an m-dimensional coordinate subspace W of $\mathbb{R}^n$ (i.e. a subspace spanned by an m-element subset of the standard basis) such that the projection of V to W is an isomorphism.

   (b) This is a nonlinear analog of (a). Let M be an m-dimensional submanifold of $\mathbb{R}^n$. Show that for any $p \in M$ there exists an m-dimensional coordinate subspace W of $\mathbb{R}^n$ such that the projection of M to W is a local diffeomorphism near p.

   (c) Hence show that M is locally a graph of a smooth map from W to the complementary coordinate subspace $\bar{W}$ (that is a coordinate subspace such that $\mathbb{R}^n = W \oplus \bar{W}$).

2. Exercise 4.32 in the textbook.

3. The *Grassmanian* $Gr(k, n)$ is defined as the set of all k-dimensional subspaces in $\mathbb{R}^n$ (for example, $Gr(1, n+1)$ is the same as $\mathbb{RP}^n$). Let also $W(k, n)$ be the set of all ordered k-tuples of linearly independent vectors in $\mathbb{R}^n$ (this is sometimes called a *non-compact Stiefel manifold*). Note that $W(k, n)$ can be identified with full rank $k \times n$ matrices and hence has a natural structure of a smooth manifold. Prove that there is a unique topology and smooth structure on $Gr(k, n)$ such that the projection $\pi\colon W(k, n) \to Gr(k, n)$, defined by $\pi(v_1, \dots, v_k) = \operatorname{span}(v_1, \dots, v_k)$, is a submersion.

   Remark: Grassmanians are discussed in Example 1.36 in the textbook. You may use their construction of charts if you want. But even if you do, you will still need to prove uniqueness and the fact that π is a submersion.