MATH534A, solutions for Problem Set 6

Problem 1.

1. Let V be an m-dimensional subspace of $\mathbb{R}^n$. Show that there exists an m-dimensional coordinate subspace W of $\mathbb{R}^n$ (i.e. a subspace spanned by an m-element subset of the standard basis) such that the projection of V to W is an isomorphism.
2. This is a nonlinear analog of (a). Let M be an m-dimensional submanifold of $\mathbb{R}^n$. Show that for any $p \in M$ there exists an m-dimensional coordinate subspace W of $\mathbb{R}^n$ such that the projection of M to W is a local diffeomorphism near p.
3. Hence show that M is locally a graph of a smooth map from W to the complementary coordinate subspace $\bar{W}$ (that is a coordinate subspace such that $\mathbb{R}^n = W \oplus \bar{W}$).

Solutions. 1. Take any basis $f_1, \ldots, f_m$ in V and consider the $m \times n$ matrix

$$\begin{pmatrix} f_1 \\ \ldots \\ f_m \end{pmatrix}.$$

The rows of this matrix are linearly independent, so it has rank m and hence admits a non-vanishing $m \times m$ minor. Denote by $i_1, \ldots, i_m$ the indices of columns forming such a minor. Then the rows of that minor are projections of the vectors $f_1, \ldots, f_m$ to the subspace W spanned by $e_{i_1}, \ldots, e_{i_m}$, where e_i's are standard basis vectors in $\mathbb{R}^n$. Since the minor is non-vanishing, it follows that those projections are linearly independent, which means that the projection of V to W is an isomorphism, as desired.

2. Applying the first part of the problem to the tangent space T_pM, we get that there exists a coordinate subspace W of $\mathbb{R}^n$ such that the projection of T_pM to W is an isomorphism. Denote the projection (of the whole space $\mathbb{R}^n$) to W by π_W. We want to show that π_W restricted to M is a local diffeomorphism. To that end, it suffices to prove that the differential of $(\pi_W)|_M$ at p is invertible. We have

$$d_p(\pi_W|_M) = (d\pi_W)|_{T_pM} = (\pi_W)|_{T_pM},$$

where we used that π_W on $\mathbb{R}^n$ is a linear map and hence coincides with its own differential. Now it suffices to notice that $(\pi_W)|_{T_pM}$ is invertible by construction, so $d_p(\pi_W|_M)$ is invertible as well, as desired.

3. By the previous part, there is an open in M neighborhood U of p such that the restriction of the projection π_W to U maps U diffeomorphically to an open subset V of W. Therefore, we have a smooth inverse $\pi_W^{-1}\colon V \to U \subset \mathbb{R}^n$. Furthermore, since $\mathbb{R}^n$, as a manifold, is a direct product of W and the complimentary subspace $\bar{W}$, we can regard π_W^{-1} as a map $V \to W \times \bar{W}$. And since π_W^{-1} is the inverse of π_W, we must have that

$$\pi_W^{-1}(x) = (x, \gamma(x)) \quad \forall\, x \in V.$$

Also note that $\gamma = \pi_{\bar{W}} \circ \pi_W^{-1}$ and hence smooth. So, the set $U \subset M$, which is the image of π_W, consists of points of the form $(x, \gamma(x))$, where $\gamma\colon W \to \bar{W}$ is smooth. By definition this means that U is the graph of γ, as desired.

Problem 2. Exercise 4.32 in the textbook: Suppose that M, N_1, and N_2 are smooth manifolds, and $\pi_1\colon M \to N_1$ and $\pi_2\colon M \to N_2$ are surjective smooth submersions that are constant on each other?s fibers. Then there exists a unique diffeomorphism $F\colon N_1 \to N_2$ such that $F \circ \pi_1 = \pi_2$.

Solution. Such F (if it exists) is unique, because it must satisfy $F(\pi_1(p)) = \pi_2(p)$ for any $p \in M$, and any element of N_1 is of the form $\pi_1(p)$ for some $p \in M$.

To prove existence, take a point $q \in N_1$. Since π_1 is surjective, there is $p \in M$ such that $\pi_1(p) = q$. Furthermore, although such p is not unique, the value $\pi_2(p)$ does not depend on the choice of p, because π_2 is constant on fibers of π_1. So, $\pi_2(p)$ only depends on q, and we set $F(q) = \pi_2(p)$. So, we get a map $F\colon N_1 \to N_2$, which, by construction, satisfies $F(\pi_1(p)) = \pi_2(p)$ for any $p \in M$, which means that $F \circ \pi_1 = \pi_2$.

Similarly, by interchanging indices 1 and 2, we construct a map $G\colon N_2 \to N_1$ such that $G \circ \pi_2 = \pi_1$. Applying F to both sides of the latter equation, we get

$$F \circ G \circ \pi_2 = F \circ \pi_1 = \pi_2,$$

and since π_2 is surjective, it follows that $F \circ G = \mathrm{id}$. Analogously, we get $G \circ F = \mathrm{id}$. So, F is a bijection with $F^{-1} = G$. To show that F is a diffeomorphism, it remains to prove that F and G are smooth, and due to symmetry it suffices to show the smoothness of F.

To prove smoothness of F, we take $p \in N_1$ and show smoothness of F at p. Since π_1 is a surjective submersion, it admits a section, i.e. a right inverse, σ defined in the neighborhood of p. In that neighborhood, we have

$$\mathrm{id} = \pi_1 \circ \sigma.$$

Multiplying this equation by F from the left, we get

$$F = F \circ \pi_1 \circ \sigma = \pi_2 \circ \sigma.$$

So, near the point p, the map F is a composition of smooth maps and hence smooth, as desired.

Problem 3. The *Grassmanian* $Gr(k, n)$ is defined as the set of all k-dimensional subspaces in $\mathbb{R}^n$ (for example, $Gr(1, n + 1)$ is the same as $\mathbb{RP}^n$). Let also $W(k, n)$ be the set of all ordered k-tuples of linearly independent vectors in $\mathbb{R}^n$ (this is sometimes called a *non-compact Stiefel manifold*). Note that $W(k, n)$ can be identified with full rank $k \times n$ matrices and hence has a natural structure of a smooth manifold. Prove that there is a unique topology and smooth structure on $Gr(k, n)$ such that the projection $\pi\colon W(k, n) \to Gr(k, n)$, defined by $\pi(v_1, \ldots, v_k) = \mathrm{span}(v_1, \ldots, v_k)$, is a submersion.

Remark: Grassmanians are discussed in Example 1.36 in the textbook. You may use their construction of charts if you want. But even if you do, you will still need to prove uniqueness and the fact that π is a submersion.

Solution. We first prove uniqueness. Uniqueness of topology follows from the fact that every surjective submersion is a quotient map, therefore the only possible topology on $Gr(k, n)$ for which π is a a submersion is the final topology with respect to π. To prove uniqueness of the smooth structure, assume that $Gr(k, n)$ admits two smooth structures for which π is a submersion. Denote $Gr(k, n)$ endowed with these two smooth structures by $Gr_1(k, n)$ and $Gr_2(k, n)$ respectively. Then we have surjective submersions $\pi_1\colon W(k, n) \to Gr_1(k, n)$ and $\pi_2\colon W(k, n) \to Gr_2(k, n)$ (these are just two copies of the map π). These two maps have the same fibers (which are just the fibers of π), so , by the previous problem, there exists a diffeomorphism $F\colon Gr_1(k, n) \to Gr_2(k, n)$ such that $F \circ \pi_1 = \pi_2$. Furthermore, since from the set-theoretic point of view we have $\pi_1 = \pi_2 = \pi$, and since π is surjective, it follows that $F = \mathrm{id}$. So, since the identity map $Gr_1(k, n) \to Gr_2(k, n)$ is a diffeomorphism, it follows that the smooth structures on $Gr_1(k, n)$ and $Gr_2(k, n)$ are the same. (In fact, this argument also proves uniqueness of the topology).

To prove existence, we take the smooth structure of $Gr(k, n)$ constructed in the textbook and show that π is a submersion for that smooth structure. The charts in $Gr(k, n)$ are parametrized by pairs of subspaces $P, Q \subset \mathbb{R}^n$ such that $\dim P = k$, $\dim Q = n - k$, and $P \cap Q = 0$. The domain of the chart corresponding to such a pair (P, Q) is the set U_Q of subspaces $V \in Gr(k, n)$ such that $V \cap Q = 0$. For such V, its projection π_P to P along Q is an isomorphism, which allows one to identify V with the graph of the map $\pi_Q \circ \pi_P^{-1}: P \to Q$. As shown in the textbook, the correspondence $\phi_{P,Q}: U_Q \to \mathrm{Hom}_{\mathbb{R}}(P, Q)$ given by $\phi_{P,Q}(V) = \pi_Q \circ \pi_P^{-1}$ is a homeomorphism, and $Gr(k, n)$ endowed with charts $(U_Q, \phi_{P,Q})$ is a smooth manifold (note that the set $\mathrm{Hom}_{\mathbb{R}}(P, Q)$ of linear maps from P to Q is a vector space, so $(U_Q, \phi_{P,Q})$ is indeed a chart). Now we show that π is a submersion with respect to this smooth manifold structure. Take a k-element set $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ and consider the subset $U_{i_1,\ldots,i_k}$ of $W(k, n)$ consisting of k-tuples $f_1, \ldots, f_k$ such that the columns with indices $i_1, \ldots, i_k$ of the matrix

$$\begin{pmatrix} f_1 \\ \cdots \\ f_k \end{pmatrix}.$$

are linearly independent. The sets of the form $U_{i_1,\ldots,i_k}$ are open and cover $W(k, n)$, so it suffices to show that π is a submersion on each of these sets. We show that π is a submersion on $U_{1,\ldots,k}$, while the proof for other subsets $U_{i_1,\ldots,i_k}$ is achieved by renumbering basis vectors in $\mathbb{R}^n$. To prove that π is a submersion on $U_{1,\ldots,k}$, note that for any $(f_1, \ldots, f_k) \in U_{1,\ldots,k}$, the corresponding subspace $V = \pi(f_1, \ldots, f_k)$ intersects trivially the subspace $\mathrm{span}(e_{k+1}, \ldots, e_n)$. So, to work with the map π on $U_{1,\ldots,k}$, we can use the chart on $Gr(k, n)$ associated with a pair of subspaces $\mathrm{span}(e_1, \ldots, e_k)$, $\mathrm{span}(e_{k+1}, \ldots, e_n)$. In this chart, the coordinate representation of π sends the matrix A (we identify elements of $W(k, n)$ with full rank $k \times n$ matrices) to the matrix $A_L^{-1} A_R$, where A_L is the submatrix of A spanned by the k leftmost columns, while A_R is spanned by the remaining columns (we identify, in a natural way, linear maps from $\mathrm{span}(e_1, \ldots, e_k)$ to $\mathrm{span}(e_{k+1}, \ldots, e_n)$ with $k \times (n-k)$ matrices). So, the map π has a smooth coordinate representation and hence smooth. Furthermore, for any $B \in U_{1,\ldots,k}$, the map

$$\sigma: Mat(k, n-k) \to Mat(k, n)$$

given by

$$\sigma(X) = (B_L, B_L X)$$

is a smooth section of the coordinate representation $A \mapsto A_L^{-1} A_R$ of π, and the image of this smooth section contains B. Therefore, the coordinate representation of π is a submersion, and so is π.